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ON PRODUCTS, POWERS, AND CONTRACTIONS OF HOMOMORPHISMS

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Abstract

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MATHEMATICS

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ON PRODUCTS, POWERS, AND CONTRACTIONS OF HOMOMORPHISMS

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A linear operator A with domain of definition $\mathfrak{D}(A)$ in a linear topological space (l.t.s.) X and range $\mathfrak{R}(A)$ in an l.t.s. Y is called a homomorphism if the image of every set open relative to $\mathfrak{D}(A)$ is open relative to $\mathfrak{R}(A)$. Numerous works have been devoted to the study of homomorphisms, especially in connection with the theory of linear closed operators. Already S. Banach established that if X and Y are spaces of type F, then a linear continuous operator A is a homomorphism if and only if $\mathfrak{R}(A)$ is closed in Y. This result was subsequently extended to a broader class of operators and to spaces of a more general type. For example, in the paper of F. E. Browder ([1]), among a large number of different results on linear operators, conditions are established under which a closed operator in certain locally convex spaces turns out to be a homomorphism.

One of the results of the present paper is a theorem stating that in arbitrary l.t.s. a homomorphism with closed null set and closed range is a closed operator (Theorem 1). In addition, certain general properties of homomorphisms are studied (Theorems 2 and 3 on the product of homomorphisms and Theorem 4 on the contraction of a homomorphism). Theorems 1–4 (independent of one another) then make it possible to formulate conditions under which the product of closed homomorphisms is a closed homomorphism (Theorem 5), and also to prove that the contractions of a homomorphism A to the sets $\mathfrak{D}(A^n)$ ($n = 2, 3, \ldots$), $\mathfrak{L}(A) = \bigcap_{n=1}^{\infty} \mathfrak{D}(A^n)$, and $\mathfrak{D}(A) \cap \mathfrak{M}(A)$, where $\mathfrak{M}(A) = \bigcap_{n=1}^{\infty} \mathfrak{R}(A^n)$, are again homomorphisms (Corollary of Theorem 4 and Theorem 6). In Lemmas 1, 2 and Theorem 7, certain properties of the sets $\mathfrak{L}(A)$ and $\mathfrak{M}(A)$ are considered (for an operator in an arbitrary set in Lemmas 1, 2, and for a closed linear operator in a Banach space in Theorem 7).

Theorem 1. *Let X and Y be l.t.s., A a homomorphism from X into Y, and $\mathfrak{Z}(A)$ the kernel of the homomorphism A. If the sets $\mathfrak{Z}(A)$ and $\mathfrak{R}(A)$ are closed, then A is a closed operator.*

Proof. It is required to prove that the graph $G(A)$ of the operator A is closed in $X \times Y$. Let $(x, y) \in \overline{G(A)}$; then $y \in \mathfrak{R}(A)$ by virtue of the closedness of $\mathfrak{R}(A)$. Let z be some preimage of the point y. Take in $\mathfrak{D}(A)$ an arbitrary neighborhood

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$u(x)$ of the point x; shifting it by $z-x$, we obtain a neighborhood $u(z)$ of the point z. Since A is a homomorphism, the set $v_u(y)=A(u(z))$ is a neighborhood of the point y in $\mathfrak{R}(A)$. Since $(x,y)\in\overline{G(A)}$, in $u(x)$ there is a point x_u such that $Ax_u=y_u\in v_u(y)$. Let z_u be some preimage in $u(z)$ of the point y_u. Consider the generalized sequences $\{x_u\}$ and $\{z_u\}$, where u runs through the partially ordered by inclusion set of all neighborhoods of the point x. Then $x_u\to x$, $z_u\to z$ and, consequently, $x_u-z_u\to x-z$. But $A(x_u-z_u)=\theta$, i.e. $x_u-z_u\in\mathfrak{Z}(A)$,

whence, by virtue of the closedness of $\mathfrak{Z}(A)$, we conclude that $x-z\in\mathfrak{Z}(A)$. Since $z\in\mathfrak{D}(A)$, it follows that $x\in\mathfrak{D}(A)$ and $Ax=Az=y$. Hence $(x,y)\in G(A)$ and, thus, $G(A)=\overline{G(A)}$.

Theorem 2. If X,Y,Z are l.t.s., A is a homomorphism from X into Y, and B is a homomorphism from Y into Z, then BA is a homomorphism from X into Z.

Proof. Let M be an open set in $\mathfrak{D}(BA)$, i.e. $M=u\cap\mathfrak{D}(BA)$, where u is open in X. It is required to prove that $BA(M)$ is open in $\mathfrak{R}(BA)$. For this we note that

$$A(u\cap\mathfrak{D}(BA))=A(u\cap\mathfrak{D}(A)\cap\mathfrak{D}(B));$$

indeed, the inclusion

$$A(u\cap\mathfrak{D}(BA))\subseteq A(u\cap\mathfrak{D}(A))\cap\mathfrak{D}(B)$$

is obvious; conversely, let

$$y\in A(u\cap\mathfrak{D}(A))\cap\mathfrak{D}(B),$$

i.e. $y=Ax$, where $x\in u\cap\mathfrak{D}(A)$, and $y\in\mathfrak{D}(B)$; then $x\in u\cap\mathfrak{D}(BA)$ and $y\in A(u\cap\mathfrak{D}(BA))$. Hence

$$BA(M)=B(A(M))=B(A(u\cap\mathfrak{D}(BA)))=B(A(u\cap\mathfrak{D}(A))\cap\mathfrak{D}(B)).$$

But by hypothesis

$$A(u\cap\mathfrak{D}(A))=v\cap\mathfrak{R}(A),$$

where v is open in Y; consequently

$$BA(M)=B(v\cap\mathfrak{R}(A)\cap\mathfrak{D}(B)).$$

Next we observe that

$$B(v\cap\mathfrak{D}(B))\cap\mathfrak{R}(BA)$$

(the inclusion $\subseteq$ is obvious); conversely, let

$$z\in B(v\cap\mathfrak{D}(B))\cap\mathfrak{R}(BA),$$

i.e. $z=By$, $y\in v\cap\mathfrak{D}(B)$, and $z\in\mathfrak{R}(BA)$; then

$$y\in v\cap\mathfrak{R}(A)\cap\mathfrak{D}(B)$$

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and

$$z \in B(v \cap \mathfrak{R}(A) \cap \mathfrak{D}(B)),$$

and since, by hypothesis,

$$B(v \cap \mathfrak{D}(B)) = w \cap \mathfrak{R}(B),$$

where w is open in Z, we have

$$BA(M) = w \cap \mathfrak{R}(B) \cap \mathfrak{R}(BA) = w \cap \mathfrak{R}(BA).$$

Corollary. If A is a homomorphism in the l.t.s. X, then A^n $(n = 2, 3, ...)$ are also homomorphisms in X.

Theorem 3. Let A be a homomorphism from the l.t.s. X into the l.t.s. Y, and B a homomorphism from Y into the l.t.s. Z. If $\mathfrak{R}(A)$ and $\mathfrak{R}(B)$ are closed respectively in Y and in Z, and

$$\gamma(B, A) \overset{\text{def}}{=} \dim(\mathfrak{Z}(B) \ominus \mathfrak{Z}(B) \cap \mathfrak{R}(A)) < \infty,$$

then $\mathfrak{R}(BA)$ is closed in Z (the symbol $\ominus$ serves to denote the algebraic complement).

The proof is based on Lemma 2 of ([2]), according to which every linear set containing all zeros of a homomorphism and closed relative to its domain is mapped by this homomorphism onto a set closed relative to its range. Since

$$\mathfrak{R}(BA) = B(\mathfrak{D}(B) \cap \mathfrak{R}(A)) = B(\mathfrak{D}(B) \cap (\mathfrak{Z}(B) + \mathfrak{R}(A))),$$

the closedness of $\mathfrak{R}(BA)$ follows, according to the lemma just mentioned, from the closedness of the set $\mathfrak{Z}(B) + \mathfrak{R}(A)$, which follows from the closedness of $\mathfrak{R}(A)$ and the condition $\gamma(B, A) < \infty$ (([3]), p. 50, Corollary 4).

Corollary. If A is a homomorphism in the l.t.s. X,

$$\gamma(A) \overset{\text{def}}{=} \dim(\mathfrak{Z}(A) \ominus \mathfrak{Z}(A) \cap \mathfrak{R}(A)) < \infty$$

and $\mathfrak{R}(A)$ is closed, then $\mathfrak{R}(A^n)$ $(n = 2, 3)$ is also closed.

Indeed, since

$$\gamma(A, A^n) = \dim(\mathfrak{Z}(A) \ominus \mathfrak{Z}(A) \cap \mathfrak{R}(A^n)) \leq \dim(\mathfrak{Z}(A) \ominus \mathfrak{Z}(A) \cap \mathfrak{R}(A)) = \gamma(A) < \infty,$$

$\mathfrak{R}(A^{n+1})$ is closed, provided $\mathfrak{R}(A^n)$ is closed.

Theorem 4. Let X and Y be l.t.s., and let A be a homomorphism from X into Y. If E is a linear set in X and $\mathfrak{Z}(A) \subseteq E \subseteq \mathfrak{D}(A)$, then the restriction of A to E is a homomorphism from X into Y.

Proof. Let u be an open set in X. The equality

$$A(u \cap E) = A(u \cap \mathfrak{D}(A)) \cap A(E)$$

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holds. Indeed, the inclusion

$$A(u \cap E) \subseteq A(u \cap \mathfrak{D}(A)) \cap A(E)$$

is obvious. Conversely, let

$$y \in A(u \cap \mathfrak{D}(A)) \cap A(E),$$

i.e. $y = Ax$, where $x \in u \cap \mathfrak{D}(A)$, and $y = Az$, where $z \in E$; then $z - x \in \mathfrak{Z}(A)$, and, since $z \in E$ and $\mathfrak{Z}(A) \subseteq E$, we have $x \in E$; consequently,

$$y \in A(u \cap E),$$

and hence

$$A(u \cap E) \supseteq A(u \cap \mathfrak{D}(A)) \cap A(E).$$

According to the hypothesis,

$$A(u \cap \mathfrak{D}(A)) = v \cap \mathfrak{R}(A),$$

where v is open in Y. But $A(E) \subseteq \mathfrak{R}(A)$, whence

$$A(u \cap E) = v \cap A(E).$$

Corollary. If A is a homomorphism in the l.t.s. X, then the restrictions of A to $\mathfrak{D}(A^n)$ $(n = 2, 3, ...)$ and to $\mathfrak{L}(A)$, and also the restrictions of A^m to $\mathfrak{D}(A^n)$ $(m < n)$ and to $\mathfrak{L}(A)$, are homomorphisms in X.

Theorem 5. Let A be a homomorphism from a separated l.t.s. X into the l.t.s. Y, and B a homomorphism from Y into the l.t.s. Z. If $\mathfrak{R}(A)$ and $\mathfrak{R}(B)$ are closed respectively in Y and Z, $\alpha(A) < \infty$ and $\alpha(B) < \infty$ (where $\alpha(\cdot) = \dim \mathfrak{Z}(\cdot)$), then BA is a closed operator.

The proof reduces to verifying the hypotheses of Theorem 1. By Theorem 2, BA is a homomorphism, and by Theorem 3, $\mathfrak{R}(BA)$ is closed in Z; moreover, $\mathfrak{Z}(BA)$ is closed, since $\alpha(BA) \leq \alpha(A) + \alpha(B) < \infty$ ((3), p. 49, Corollary 1).

Corollary. *If A is a homomorphism in a separated l.t.s. X, $\mathfrak{R}(A)$ is closed and $\alpha(A) < \infty$, then the operators A^n $(n = 1, 2, ...)$ are closed.*

The theorem proved does not extend to the case when $\alpha(A) = \infty$, as can be seen from the following example. Let H be an infinite-dimensional Hilbert space, T a one-to-one linear discontinuous operator defined on H, with nonclosed range $\mathfrak{R}(T)$ in H. Then T^{-1} will be a homomorphism in H. Consider the linear operator A, defined on the nonclosed set $\mathfrak{R}(T) \times H$ in the Hilbert space $H \times H$ by the equalities $A(x, \theta) = (T^{-1}x, \theta)$ $(x \in \mathfrak{R}(T))$, $A(\theta, y) = (\theta, \theta)$ $(y \in H)$. The operator A is a homomorphism in $H \times H$; by Theorem 1, A is a closed operator. However, A^2 is not a closed operator, since it is a bounded operator with nonclosed domain of definition $(A^2 = 0,\ \mathfrak{D}(A^2) = \mathfrak{R}(T) \times H)$.

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Theorem 6. *Let X be a locally convex space, A a homomorphism in X, $\mathfrak{R}(A)$ closed, and $\gamma(A) < \infty$. Then the restriction of A to $\mathfrak{D}(A) \cap \mathfrak{M}(A)$ is a homomorphism in X.*

Proof. By the corollary to Theorem 3, the set $\mathfrak{M}(A)$ is closed in X, and according to (4) (Theorem 1)

$$A(\mathfrak{D}(A) \cap \mathfrak{M}(A)) = \mathfrak{M}(A).$$

Let

$$\mathfrak{F}(A) = \mathfrak{M}(A) \oplus (\mathfrak{Z}(A) \dotplus \mathfrak{D}(A) \cap \mathfrak{M}(A)).$$

Since $\mathfrak{Z}(A) = \mathfrak{D}(A) \cap \mathfrak{F}(A)$, by Theorem 4 the restriction of A to $\mathfrak{D}(A) \cap \mathfrak{F}(A)$ is a homomorphism in X. Denote by P the continuous projection operator of $\mathfrak{F}(A)$ onto $\mathfrak{M}(A)$ (such an operator exists, since X is locally convex and $\gamma(A) < \infty$). Then, for any u open in X, the set $P^{-1}(u \cap \mathfrak{M}(A))$ is open in $\mathfrak{F}(A)$, i.e. $P^{-1}(u \cap \mathfrak{M}(A)) = v \cap \mathfrak{F}(A)$, where v is open in X. But

$$A(u \cap \mathfrak{D}(A) \cap \mathfrak{M}(A)) = A(\mathfrak{D}(A) \cap P^{-1}(u \cap \mathfrak{M}(A))) = A(v \cap \mathfrak{D}(A) \cap \mathfrak{F}(A)),$$

and since the restriction of A to $\mathfrak{D}(A) \cap \mathfrak{F}(A)$ is a homomorphism, $A(u \cap \mathfrak{D}(A) \cap \mathfrak{M}(A))$ is open in $A(\mathfrak{D}(A) \cap \mathfrak{M}(A))$.

Corollary. *Under the conditions of Theorem 6, on the basis of Theorem 4 we conclude that the restriction of A to $\mathfrak{L}(A) \cap \mathfrak{M}(A)$ is a homomorphism in X.*

In the following two theorems the A-invariance of the sets $\mathfrak{L}(A)$ and $\mathfrak{L}(A) \cap \mathfrak{M}(A)$ is proved, which makes it possible to consider powers of the restrictions of the operator A to these sets.

Lemma 1. *Let X be an arbitrary set, A an operator acting in X, and*

$$\mathfrak{L}(A) = \bigcap_{n=1}^{\infty} \mathfrak{D}(A^n).$$

Then the inclusions $A(\mathfrak{L}(A)) \subseteq \mathfrak{L}(A)$ and $A^{-1}(\mathfrak{L}(A)) \subseteq \mathfrak{L}(A)$ hold, where $A^{-1}(\mathfrak{L}(A))$ is the full preimage of the set $\mathfrak{L}(A)$.

Proof. Let $x \in \mathfrak{L}(A)$; then $x \in \mathfrak{D}(A^{n+1}) = A^{-1}(\mathfrak{D}(A^n))$ and $Ax \in \mathfrak{D}(A^n)$, whence, in view of the arbitrariness of n, $Ax \in \mathfrak{L}(A)$, so that $A(\mathfrak{L}(A)) \subseteq \mathfrak{L}(A)$. Let $x \in A^{-1}(\mathfrak{L}(A))$; then $x \in A^{-1}(\mathfrak{D}(A^{n-1})) = \mathfrak{D}(A^n)$; since n is arbitrary, $x \in \mathfrak{L}(A)$, whence

$$A^{-1}(\mathfrak{L}(A)) \subseteq \mathfrak{L}(A).$$

It is obvious that the inclusions proved in this lemma imply the equality

$$A^{-1}(\mathfrak{L}(A)) = \mathfrak{L}(A).$$

Lemma 2. *Let X, A, $\mathfrak{L}(A)$ have the same meaning as in Lemma 1, and*

$$\mathfrak{M}(A) = \bigcap_{n=1}^{\infty} \mathfrak{R}(A^n),$$

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where $\mathfrak{R}(A^n)$ *is the range of the operator* A^n*. If*

$$A(\mathfrak{D}(A) \cap \mathfrak{M}(A)) = \mathfrak{M}(A),$$

then

$$A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) = \mathfrak{L}(A) \cap \mathfrak{M}(A).$$

Proof. Since $A(\mathfrak{L}(A)) \subseteq \mathfrak{L}(A)$, we have $A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) \subseteq \mathfrak{L}(A)$; moreover,

$$A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) \subseteq A(\mathfrak{D}(A) \cap \mathfrak{M}(A)) = \mathfrak{M}(A).$$

Consequently,

$$A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) \subseteq \mathfrak{L}(A) \cap \mathfrak{M}(A).$$

On the other hand, if $x \in \mathfrak{L}(A) \cap \mathfrak{M}(A)$, then, by the condition of the theorem, $x = Ay$, where $y \in \mathfrak{D}(A) \cap \mathfrak{M}(A)$; at the same time, by Lemma 1, $y \in \mathfrak{L}(A)$. Thus,

$$y \in \mathfrak{D}(A) \cap \mathfrak{M}(A) \cap \mathfrak{L}(A) =$$

$$= \mathfrak{L}(A) \cap \mathfrak{M}(A).$$

Hence, by virtue of the arbitrariness of $x \in \mathfrak{L}(A) \cap \mathfrak{M}(A)$, it follows that

$$\mathfrak{L}(A) \cap \mathfrak{M}(A) \subseteq (\mathfrak{L}(A) \cap \mathfrak{M}(A)).$$

Let us note that, on the basis of Lemma 1 and the corollary to Theorem 2, the restrictions to $\mathfrak{L}(A)$ of a homomorphism A into X, as well as the powers of this restriction, are homomorphisms into $\mathfrak{L}(A)$. Analogously, under the conditions of Theorem 6, on the basis of Lemma 2 we conclude that the restriction of A to $\mathfrak{L}(A) \cap \mathfrak{M}(A)$ and the powers of this restriction are homomorphisms into $\mathfrak{L}(A) \cap \mathfrak{M}(A)$.

When considering the restriction of A to $\mathfrak{L}(A)$, the question arises whether the set $\mathfrak{L}(A)$ is not too small—for example, whether it reduces to a single zero point. In the case where X is a Banach space, A is a closed linear operator in X, $\mathfrak{R}(A) = X$, and $\mathfrak{D}(A)$ is dense in X, an answer to this question is given by a theorem of D. A. Raikov, asserting that under these conditions $\mathfrak{L}(A)$ is also dense in X. Wishing to consider the case $\mathfrak{R}(A) \neq X$ (under the same remaining conditions), we shall prove the following theorem.

Theorem 7. *Let* X *be a Banach space, and let* A *be a closed linear operator with closed range* $\mathfrak{R}(A)$ *and with* $\gamma(A)$ *finite. Then, if* $\mathfrak{D}(A)$ *is dense in the set* $\mathfrak{M}(A)$*, then*

$$A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) = \mathfrak{L}(A) \cap \mathfrak{M}(A)$$

and $\mathfrak{L}(A)$ *is dense in* $\mathfrak{M}(A)$*.*

Proof. By virtue of the assumptions made (without using the fact that $\overline{\mathfrak{D}(A)} \subseteq \mathfrak{M}(A)$), the set $\mathfrak{M}(A)$ is closed in X. Consider the restriction $\bar{A}$ of the operator

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A to $\mathfrak{D}(A) \cap \mathfrak{M}(A)$. This will be a closed operator mapping the set $\mathfrak{D}(A) \cap \mathfrak{M}(A)$, dense in $\mathfrak{M}(A)$, onto the whole space $\mathfrak{M}(A)$. Thus the hypothesis of Lemma 2 is fulfilled, and hence

$$A(\mathfrak{L}(A) \cap \mathfrak{M}(A)) = \mathfrak{L}(A) \cap \mathfrak{M}(A).$$

Further, by virtue of the above-mentioned result of D. A. Raikov, we conclude that the set $\mathfrak{L}(\bar{A})$ is dense in $\mathfrak{M}(A)$. But

$$\mathfrak{L}(\bar{A}) \subseteq \mathfrak{L}(A),$$

for

$$\mathfrak{D}(\bar{A}^n) \subseteq \mathfrak{D}(A^n) \qquad (n = 1, 2, ...).$$

Consequently, $\mathfrak{L}(A)$ is dense in $\mathfrak{M}(A)$.

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