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ON THE STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN THE PROBLEM OF AUTONOMOUS DETERMINATION OF THE COORDINATES OF A MOVING OBJECT

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1. Determination of the coordinates of the location of a moving object in an inertial navigation system with a platform stabilized in the horizon is carried out, as is known ([1,2]), by solving nonlinear differential equations of the form

$$-\frac{d\varphi}{dt}\cos\chi + \left(U + \frac{d\lambda}{dt}\right)\cos\varphi\sin\chi = \omega_x,$$

$$\frac{d\varphi}{dt}\sin\chi + \left(U + \frac{d\lambda}{dt}\right)\cos\varphi\cos\chi = \omega_y, \qquad (1)$$

$$\left(U + \frac{d\lambda}{dt}\right)\sin\varphi + \frac{d\chi}{dt} = \omega_z,$$

where φ, λ are, respectively, the latitude and longitude of the location of the moving object on which the inertial navigation system is placed; U is the angular velocity of the daily rotation of the Earth; χ is the angle characterizing the orientation of the platform relative to the geographic coordinate system; $\omega_x, \omega_y, \omega_z$ are the projections of the vector of the angular velocity of the stabilized platform during the motion of the object over the terrestrial sphere onto the axes xyz of the coordinate system rigidly connected with the platform.

Putting in (1) ([3])

$$\lambda + Ut = \psi, \qquad \pi/2 - \varphi = \theta \qquad (2)$$

and introducing the Rodrigues-Hamilton parameters ([4,5]) by means of the relations

$$r_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\chi}{2}, \qquad r_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\chi}{2},$$

$$r_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\chi}{2}, \qquad r_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\chi}{2}, \tag{3}$$

according to which

$$r_0^2 + r_1^2 + r_2^2 + r_3^2 = 1, \tag{4}$$

we arrive at a system of linear differential equations with respect to the parameters r_s $(s = 0, 1, 2, 3)$ of the form

$$2\,dr_0/dt = -\omega_x r_1 - \omega_y r_2 - \omega_z r_3, \qquad 2\,dr_2/dt = \omega_x r_3 + \omega_y r_0 - \omega_z r_1,$$

$$2\,dr_1/dt = \omega_x r_0 - \omega_y r_3 + \omega_z r_2, \qquad 2\,dr_3/dt = -\omega_x r_2 + \omega_y r_1 + \omega_z r_0. \tag{5}$$

2. Let us turn to the study of Lyapunov stability of system (5) for an arbitrary set of continuous functions $\omega_x(t), \omega_y(t), \omega_z(t)$. Suppose that in the perturbed motion

$$r_s = r_s^* + \xi_s, \tag{6}$$

where r_s^* are the Rodrigues-Hamilton parameters for some particular so-

solution of system (5) corresponding to the unperturbed motion; ξ_s are, respectively, small perturbations.

By virtue of (6), the equations of the perturbed motion will have the form

$$\begin{aligned} &2d\xi_0/dt = -\omega_x\xi_1 - \omega_y\xi_2 - \omega_z\xi_3, \qquad &&2d\xi_2/dt = \omega_x\xi_3 + \omega_y\xi_0 - \omega_z\xi_1,\\ &2d\xi_1/dt = \omega_x\xi_0 - \omega_y\xi_3 + \omega_z\xi_2, &&2d\xi_3/dt = -\omega_x\xi_2 + \omega_y\xi_1 + \omega_z\xi_0. \end{aligned} \tag{7}$$

System (7) coincides with system (5) up to the notation.

As a Lyapunov function it is natural to take the obvious integral of the equations of the perturbed motion (7), namely:

$$V \equiv \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2. \tag{8}$$

The function V is positive definite. Its total time derivative by virtue of system (7) is identically equal to zero; consequently, by Lyapunov' s theorem (6), the unperturbed motion corresponding to r_s^* will be stable.

3. To prove Lyapunov stability of system (1), we shall use the formulas for the inverse transition to the original variables. This transition, carried out according to (3) by means of the Rodrigues-Hamilton parameters, generally speaking requires special consideration because of the multivaluedness of the indicated functions. It is not difficult to show that the transition to the required solutions of system (1) is single-valued in the domain

$$-\pi/2+\varepsilon \le \varphi \le \pi/2-\varepsilon, \tag{9}$$

where $\varepsilon > 0$ is arbitrarily small.

We perform a change of variables, setting

$$x = (\psi-\chi)/2, \qquad y = (\psi+\chi)/2. \tag{10}$$

From relations (3), using inequality (9), we obtain

$$\begin{aligned} \varphi &= \arcsin(1-2r_1^2-2r_2^2), \\ x &= \operatorname{arc\,tg}(r_2/r_1)+m\pi, \\ y &= \operatorname{arc\,tg}(r_3/r_0)+n\pi, \end{aligned} \tag{11}$$

where m and n are quantities taking integer values and changing by one whenever the values of the functions r_2/r_1 and r_3/r_0 undergo a discontinuity. The functions x and y defined in this way are continuous. This assertion follows from the circumstance that the ratios r_2/r_1 and r_3/r_0 cannot become indeterminate by virtue of the inequalities following from expressions (3), (4), (9),

$$r_1^2+r_2^2 \ge \sin^2\varepsilon/2 > 0, \qquad r_0^2+r_3^2 \ge \tfrac{1}{2}(1-\cos\varepsilon) > 0. \tag{12}$$

As for the function φ, defined by the first of equalities (11), it is continuous by virtue of the inequality following from (4), namely:

$$-1 \le 1-2r_1^2-2r_2^2 \le 1. \tag{13}$$

Let us assume that in the perturbed motion corresponding to (6),

$$\varphi = \varphi^*+\Delta\varphi, \qquad x = x^*+\Delta x, \qquad y = y^*+\Delta y, \tag{14}$$

where the increments $\Delta\varphi, \Delta x$ and Δy are to be estimated.

Taking (6) and (14) into account, expressions (11) are reduced to the form

$$\begin{aligned} \varphi^* + \Delta\varphi &= \arcsin[1 - 2(r_1^* + \xi_1)^2 - 2(r_2^* + \xi_2)^2], \\ x^* + \Delta x &= \operatorname{arc\,tg}[(r_2^* + \xi_2)/(r_1^* + \xi_1)] + m\pi, \\ y^* + \Delta y &= \operatorname{arc\,tg}[(r_3^* + \xi_3)/r_0^* + \xi_0] + n\pi. \end{aligned} \tag{15}$$

By virtue of the continuity of the functions φ, x, and y, the values $\Delta\varphi, \Delta x, \Delta y$ can be made arbitrarily small if the functions ξ_s are sufficiently small. The latter, by virtue of the stability of system (5), are determined by the choice of the initial conditions $\xi_s(0)$. In turn, from (14), (10), and (2) it follows that

$$\Delta\lambda = \Delta x + \Delta y, \qquad \Delta\chi = \Delta y - \Delta x. \tag{16}$$

Thus, on the basis of the analysis presented, one may draw the general conclusion that, for any continuous functions $\omega_x(t), \omega_y(t)$, and $\omega_z(t)$, system (1) is Lyapunov stable.

It may be noted that this conclusion, which agrees with the results obtained in ([7],[8]), can be extended to any computing device of an inertial-navigation system whose equations lead to Darboux' s problem of determining the coordinates of a body rotating about a fixed point from given projections of its angular velocity ([4]).

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Note: Figure translations are in progress. See original paper for figures.