NONLINEAR EQUATIONS WITH POTENTIAL AND MONOTONE OPERATORS

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NONLINEAR EQUATIONS WITH POTENTIAL AND MONOTONE OPERATORS

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Nonlinear equations with potential and monotone operators were considered by us in a number of works (for example, in ([1–3])). Later such equations were studied in the works of J. L. Lions ([4]), F. Browder (see, for example, ([5]), where a bibliography is given), G. Minty ([6]), and in the works of other authors. In the present paper we study such equations in separable locally convex spaces and establish new propositions for them.

1. Let E be a real reflexive locally convex space contained in some real Hilbert space H. Everywhere we shall assume that condition (α) of ([7]) is satisfied, i.e. $E \subset H$, H is contained in the strong dual space E' and is dense in it, the topologies of the spaces E and H, and also of H and E', respectively, are compatible, and the bilinear functional $\langle x, y\rangle$, where $x \in E$, $y \in E'$, coincides for $y \in H$ with the scalar product (x, y) in H. Examples of such spaces E, H, and E' are indicated in ([7]). They include, in particular, the Lebesgue spaces $E = L^p(D)$, $H = L^2(D)$, where D is a set of finite measure in s-dimensional Euclidean space and $p > 2$. Other important examples of such spaces are rigged Hilbert spaces ([8]).

2. Consider the equation

$$x = BF(x), \qquad x \in E, \tag{1}$$

where F is a nonlinear operator from E into E' and B is a linear bounded operator from E' into E.

Let B_H be the restriction of the operator B to H. Suppose that B_H is a self-adjoint positive operator in H. Then ([7]) the positive square root $A = B_H^{1/2}$ is bounded from H into E, and the adjoint operator A' acts and is bounded from E' into H, with $B = AA'$.

In this case consider the equation

$$y = A'F(Ay), \qquad y \in H. \tag{2}$$

Equations (1) and (2) are called equivalent if the sets of their solutions are equivalent.

Lemma 1. *Equations (1) and (2) are equivalent; moreover, every solution* y *of equation (2) leads to a solution* $x = Ay$ *of equation (1), and every solution* x *of equation (1) leads to a solution* $y = A'F(x)$ *of equation (2).*

3. Let B_H be an indefinite operator in H,

$$B_H^+ = \tfrac{1}{2}(B_H + |B_H|), \qquad B_H^- = \tfrac{1}{2}(B_H - |B_H|),$$

$$A = (B_H^+)^{1/2} - |B_H^-|^{1/2}, \qquad T = (B_H^+)^{1/2} + |B_H^-|^{1/2}.$$

Then ([7]) the operator A is bounded from H into E, the operator T' is bounded from E' into H, and $B = AT'$,

Let us consider in this case the equation

$$y = T'F(Ay), \qquad y \in H. \tag{3}$$

Lemma 2. *Equations (1) and (3) are equivalent; moreover, every solution* y *of equation (3) leads to a solution* $x = Ay$ *of equation (1), and every solution* x *of equation (1) leads to a solution* $y = T'F(x)$ *of equation (3).*

4. Let B_H be a self-adjoint positive operator in H, and let x_1, x_2 be distinct solutions of equation (1). They correspond to distinct solutions y_1 and y_2 of equation (2), so that

$$0 < (y_1 - y_2, y_1 - y_2) = (y_1 - y_2, A'[F(Ay_1) - F(Ay_2)]) =$$

$$= \langle x_1 - x_2, F(x_1) - F(x_2)\rangle.$$

An operator F from E into E' (not necessarily linear) satisfying the condition

$$\langle x_1 - x_2, F(x_1) - F(x_2)\rangle \geq 0 \tag{4}$$

for arbitrary $x_1, x_2 \in E$ was called in ([3]) positive definite, and in ([9]) monotone. Thus, on the set of solutions of equation (1), the operator F is strictly monotone, i.e. in (4) equality is excluded when $x_1 \neq x_2$. Hence the following assertion follows. If the operator $(-1)F$ is monotone, i.e.

$$\langle x_1 - x_2, F(x_1) - F(x_2)\rangle \leq 0, \tag{5}$$

then equation (1) cannot have more than one solution.

Theorem 1. *Suppose the following conditions are satisfied: 1) a continuous operator $F(x)$, acting from E into E', satisfies inequality (5); 2) B_H is a self-adjoint positive operator in H. Then equation (1) has a unique solution, and it belongs to the space E.*

Let us note that this theorem is an analogue of Theorem 2.1 from ([2]), established by us for Lebesgue spaces, and the conditions of Theorem 2.1 from ([2]) ensure the uniqueness of the solution established there.

5. Here we shall assume that B_H is a quasi-positive operator ([1]), $F(x)$ is a potential operator ([1,10]), continuous along every segment of the space E (hemicontinuous ([11])), and the potential of the operator $F(x)$, i.e. $f(x)$, is continuous on every bounded set $C \subset E$ endowed with the topology τ induced in C by the space E ([4]).

Theorem 2. *Suppose the following conditions are satisfied: 1) B is a linear bounded operator from E' into E, whose restriction B_H is a quasi-positive operator in H; 2) $F(x)$ is a potential monotone and hemicontinuous operator in E, whose potential $f(x)$ is continuous on every bounded set in E with respect to the topology τ and satisfies the inequality*

$$f(x) \geq \lambda_1(x,x) + b(x,x)^{\alpha/2} + c, \qquad x \in E,$$

where λ_1 is the largest (positive) characteristic number of the operator B_H, $0 \leq \alpha < 2$, and b and c are arbitrary negative numbers. Then equation (1) has a solution belonging to the space E.

The proof is carried out according to the scheme indicated in ([1,2]), and uses the following proposition.

Lemma 3. *In order that a real functional $f(x)$, defined on a locally convex space, be sequentially weakly lower semicontinuous, it is necessary and sufficient that the following condition be satisfied: for any real number c, the set*

$$E_c = \{x : f(x) \leq c\}$$

is sequentially weakly closed.

The property is also used that if a functional defined in a vector space (not necessarily a Banach space) is Gateaux differentiable, then for its convexity (strict convexity) it is sufficient that its gradient be a monotone (strictly monotone) operator.

If the potentiality of the operator F is abandoned, we arrive at the following proposition.

Theorem 3. *Suppose that the following conditions are satisfied: 1) B_H is a quasi-positive operator in H; 2) the continuous operator F from E into E' satisfies the condition*

$$\langle h, F(x+h) - F(x) \rangle \geqslant 2\lambda_1(h,h), \qquad h \in E,$$

where λ_1 *is the largest (positive) characteristic number of the operator* B_H*. Then equation (1) has a unique solution in* H*.*

6. Here we shall consider the operator $\Gamma = BF$ under the assumption that B_H is a quasi-positive operator in H and F is a potential operator satisfying the conditions: it is strictly positive, i.e. $\langle x, F(x) \rangle > 0$ for $x \neq 0$, and $F(0) = 0$. In this case various propositions concerning the eigenvectors of the operator Γ are valid. We give one such proposition.

Theorem 4. *Let* F *be a strictly positive operator,* $F(0) = 0$*, and let* B *be a bounded operator from* E' *into* E *such that* B_H *is a quasi-positive operator in* H*. Then there exists an* $r > 0$ *such that, whatever hyperboloid in* H*,* $((u)) = c < r$*, generated by the operator* B_H*, is taken, there exist in* E *two eigenvectors of the operator* Γ*, and they are representable in the form*

$$x_c^{(1)} = Au_c^{(1)}, \qquad x_c^{(2)} = Tu_c^{(2)}, \qquad ((u_c^{(1)})) = ((u_c^{(2)})) = c.$$

These eigenvectors correspond respectively to the positive eigenvalues

$$\mu_c^{(i)} = c^{-2} \langle x_c^{(i)}, F(x_c^{(i)}) \rangle, \qquad i = 1, 2.$$

Among the eigenvectors $x_c^{(1)}$ (and $x_c^{(2)}$) there is a continuum of such vectors whose norms in H are less than an arbitrary positive number.

Let us also note that if, in the hypotheses of Theorem 4, one additionally requires that the potential $f(x)$ of the operator F satisfy, for vectors $v \in V \subset H$ (where V is the cone ([1]) generated by the operator B_H), the conditions

$$\lim_{\|v\| \to \infty} f(Av) = +\infty, \qquad \lim_{\|v\| \to +\infty} f(Tv) = +\infty,$$

then the assertion of Theorem 4 is valid for every hyperboloid; moreover, among the eigenvectors of the operator Γ there is also a continuum of such vectors whose norms in H are greater than any positive number.

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* *Note added in proof.* The assertion of the theorem remains valid if $2\lambda_1$ is replaced in the inequality by $(1+\alpha)/m$, $\alpha > 0$, if $(0, m)$ contains no spectral points.

Note: Figure translations are in progress. See original paper for figures.