ON THE ASYMPTOTIC BEHAVIOR OF AN INTEGRAL OF TRIGONOMETRIC POLYNOMIALS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.521.8

MATHEMATICS

A. A. ZAKHAROV

ON THE ASYMPTOTIC BEHAVIOR OF AN INTEGRAL OF TRIGONOMETRIC POLYNOMIALS

(Presented by Academician I. M. Vinogradov on 20 XI 1967)

Let

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) \qquad (n = 1, 2, ...) \tag{1}$$

be a given sequence of trigonometric polynomials, and let $\{\rho_n(t)\}$ $(n = 1, 2, ...)$ be a sequence of functions of bounded variation, given on the interval $[-\pi, \pi]$ and satisfying on it the conditions

$$1) \quad \bigvee_{-\pi}^{\pi} \rho_n(t) = 1 \quad (n = 1, 2, ...);$$

$$2) \quad \bigvee_{-\alpha_n-\varepsilon_n}^{-\alpha_n+\varepsilon_n} \rho_n(t) + \bigvee_{\alpha_n-\varepsilon_n}^{\alpha_n+\varepsilon_n} \rho_n(t) \geq 1 - \frac{c_1}{n}, \qquad \left(0 \leq \varepsilon_n \leq \frac{c_2}{n};\ 0 \leq \alpha_n \leq \frac{\pi}{2}\right), \tag{2}$$

where c_1 and c_2 are nonnegative constants independent of n.

In the present note we establish the asymptotic behavior of the integrals

$$\mathcal{L}_n(T, \rho) = \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \left\{ \frac{a_0}{2} + \sum_{k=1}^{n-1} [a_k \cos k(t + x) + b_k \sin k(t + x)] \right\} d\rho_n(t) \right| dx \tag{3}$$

with a remainder term of the form

$$|a_0| + \sum_{k=1}^{n-1} \frac{k(n-k)}{n} (|\Delta^2 a_{k-1}| + |\Delta^2 b_{k-1}|),$$

where

$$b_0 = a_n = b_n = 0.$$

The integrals (3) are a generalization of the norms of trigonometric polynomials in the metric L. Such a generalization for even polynomials with convex or concave coefficients and one point of concentration of variation of the functions $\rho_n(t)$ was proposed by V. F. Vlasov and A. F. Timan ([1]). The presence of two points of concentration of variation completely includes in (3) the Lebesgue constants of the Bernstein–Rogozinskii summation method.

Theorem 1. If the sequence of functions $\{\rho_n(t)\}$ satisfies conditions (2), then, uniformly in n, α_n, and $T_n(x)$, the following asymptotic equality holds:

$$\mathcal{L}_n(T,\rho) = \frac{4}{\pi} \sum_{k=1}^{\left[\frac{n}{1+n\alpha_n}\right]} \frac{1}{k}\,\xi\left(b_k \int_{-\pi}^{\pi} d\rho_n(t), \left|\int_{-\pi}^{\pi} e^{int}\, d\rho_n(t)\right| \sqrt{a_{n-k}^2 + b_{n-k}^2}\right) +$$

$$+\frac{4}{\pi} \sum_{k=\left[\frac{n}{1+n\alpha_n}\right]+1}^{n} \frac{1}{k}\,\xi\left(b_k \int_{-\pi}^{0} d\rho_n(t), \left|\int_{-\pi}^{0} e^{int}\, d\rho_n(t)\right| \sqrt{a_{n-k}^2 + b_{n-k}^2}\right) +$$

$$+\frac{4}{\pi} \sum_{k=\left[\frac{n}{1+n\alpha_n}\right]+1}^{n} \frac{1}{k}\,\xi\left(b_k \int_{0}^{\pi} d\rho_n(t), \left|\int_{0}^{\pi} e^{int}\, d\rho_n(t)\right| \sqrt{a_{n-k}^2 + b_{n-k}^2}\right) +$$

$$+O\left[|a_0| + \sum_{k=1}^{n-1} \frac{k(n-k)}{n} \left(|\Delta^2 a_{k-1}| + |\Delta^2 b_{k-1}|\right)\right], \tag{4}$$

where

$$\xi(u,v) = \frac{1}{4}\int_{-\pi}^{\pi} |u + v\sin t|\, dt.$$

Theorem 1 is formulated for the case of a symmetric arrangement of the points of concentration of variation and $0 \le \alpha_n \le \pi/2$. However, it is not difficult to show that the case of an arbitrary arrangement on $[-\pi,\pi]$ of two points of concentration of variation reduces to the one studied, with accuracy up to the remainder term of formula (4).

With the aid of Theorem 1, under various particular assumptions concerning the sequences of polynomials $\{T_n(x)\}$, functions $\{\rho_n(t)\}$, and numbers $\{\alpha_n\}$, one obtains theorems on the uniform summability of Fourier series in C of S. M. Nikol' skii ([3]), A. F. Timan ([7]), A. V. Efimov ([2]) by the method of a triangular

Λ-matrix, and theorems of A. F. Timan and I. M. Ganzburg ([8]) and G. A. Fomin ([9]) (see Theorem 5), relating to summability methods similar to the Bernstein-Rogozinskii method. Theorem 1 also gives asymptotic formulas for the Lebesgue constants and an estimate of the norms of trigonometric polynomials in the metric L. Thus, from (4) one can obtain results related to this topic of S. B. Stechkin ([4]), A. F. Timan ([7]), and S. A. Telyakovskii ([5,6]).

We now note a number of corollaries.

Let $b_k = 0$ $(k = 1, 2, \dots, n-1)$; then from Theorem 1 we obtain

Corollary 1. *If the sequence of functions* $\{\rho_n(t)\}$ *satisfies conditions (2), then for any sequence of numbers* $a_0, a_1, \dots, a_{n-1}$, $a_n = 0$, *uniformly in* n *and* α_n, *the following asymptotic equality holds*

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \left\{ \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos k(t+x) \right\} d\rho_n(t) \right| dx$$

$$= \frac{4}{\pi} \left| \int_{-\pi}^{\pi} e^{int}\, d\rho_n(t) \right| \sum_{k=1}^{\left[\frac{n}{1+n\alpha_n}\right]} \frac{|a_{n-k}|}{k} + \frac{4}{\pi} \left\{ \left| \int_{-\pi}^{0} e^{int}\, d\rho_n(t) \right| + \left| \int_{0}^{\pi} e^{int}\, d\rho_n(t) \right| \right\}$$

$$\times \sum_{k=\left[\frac{n}{1+n\alpha_n}\right]+1}^{n} \frac{|a_{n-k}|}{k} + O\left(|a_0| + \sum_{k=1}^{n-1} \frac{k(n-k)}{n} \left|\Delta^2 a_{k-1}\right| \right). \qquad (5)$$

If

$$\rho_n(t) = \begin{cases} 0, & \text{for } -\pi \le t \le -\alpha_n, \\ \frac{1}{2}, & \text{for } -\alpha_n < t \le \alpha_n, \\ 1, & \text{for } \alpha_n < t \le \pi, \end{cases} \qquad (6)$$

then from Corollary 1 we obtain

Corollary 2*. *For any sequence of numbers* $a_0, a_1, \dots, a_{n-1}$, $a_n = 0$, *uniformly in* n *and* α_n $(0 \le \alpha_n \le \pi/2)$, *the following asymptotic equality holds*

$$\int_{-\pi}^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos k\alpha_n \cos kx \right| dx = \frac{4}{\pi} |\cos n\alpha_n| \sum_{k=1}^{\left[\frac{n}{1+n\alpha_n}\right]} \frac{|a_{n-k}|}{k} +$$

Corollary 2 with proof was published in the author' s paper ([10]).

$$+ \frac{4}{\pi} \sum_{k=\left[\frac{n}{1+n\alpha_n}\right]+1}^{n} \frac{|a_{n-k}|}{k} + O\left(|a_0| + \sum_{k=1}^{n-1} \frac{k(n-k)}{n} \left|\Delta^2 a_{k-1}\right| \right). \qquad (7)$$

Let $a_k = 0$ $(k = 0, 1, \dots, n-1)$, $b_k = 1$ $(k = 1, 2, \dots, n-1;\ b_n = 0)$, and let $\rho_n(t)$ be defined by (6); then from Theorem 1 we obtain

Corollary 3. Uniformly in n and α_n $(0 \le \alpha_n \le \pi/2)$ the asymptotic equality holds

$$\int_{-\pi}^{\pi} \left| \sum_{k=1}^{n-1} \cos k\alpha_n \, \sin kx \right| dx = 2 \ln n + O(1). \tag{8}$$

Theorem 2. If the sequence of functions $\{\rho_n(t)\}$ satisfies conditions (2), then, for any sequence of numbers $a_0, a_1, \dots, a_{n-1}$, $a_n = 0$, uniformly in n and α_n the asymptotic equality holds

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \left\{ \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos k(t+x) \right\} d\rho_n(t) \right| dx =$$

$$= 2 \left\{ \left| \int_{-\pi}^{0} e^{int} \, d\rho_n(t) \right| + \left| \int_{0}^{\pi} e^{int} \, d\rho_n(t) \right| \right\} \int_{0}^{\alpha_n} \left| \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos kx \right| dx$$

$$+ 2 \left| \int_{-\pi}^{\pi} e^{int} \, d\rho_n(t) \right| \int_{\alpha_n}^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos kx \right| dx$$

$$+ O\left(|a_0| + \sum_{k=1}^{n-1} \frac{k(n-k)}{n} |\Delta^2 a_{k-1}| \right). \tag{9}$$

Theorem 2 follows immediately from Corollary 1 and Theorem 2 of the work of S. A. Telyakovskii ([6]). In the case when the sequence of numbers $\{a_k\}$ $(k = 0, 1, \dots, n-1;\ a_n = 0)$ is convex or concave and $\alpha_n = 0$, formula (9) coincides with the result of V. F. Vlasov and A. F. Timan ([1]).

The author expresses deep gratitude to S. A. Telyakovskii for supervising the work.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
13 XI 1967

CITED LITERATURE

[1] V. F. Vlasov, A. V. Timan, DAN, 63, 619 (1948).
[2] A. V. Efimov, Izv. AN SSSR, ser. matem., 24, 743 (1960).
[3] S. M. Nikol' skii, Izv. AN SSSR, ser. matem., 12, 253 (1948).
[4] S. B. Stechkin, DAN, 75, 165 (1950).
[5] S. A. Telyakovskii, Tr. Mat. inst. im. V. A. Steklova AN SSSR, 62, 61 (1961).
[6] S. A. Telyakovskii, Izv. AN SSSR, ser. matem., 27, 253 (1963).
[7] A. F. Timan, DAN, 101, 221 (1955).

[8] A. F. Timan, I. M. Ganzburg, DAN, 63, 619 (1948).
[9] G. A. Fomin, Izv. AN SSSR, ser. matem., 31, 335 (1967).
[10] A. A. Zakharov, Sibirsk. matem. zhurn., 9, 67 (1968).

Note: Figure translations are in progress. See original paper for figures.