

# ON A CLASS OF PSEU- DODIFFERENTIAL OPERATORS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON A CLASS OF PSEUDODIFFERENTIAL OPERATORS**

*(Presented by Academician I. G. Petrovskii, February 28, 1968)*

1. In this paper we continue the study of pseudodifferential operators of principal type, begun in <sup>(1-3)</sup>. As L. Hörmander showed (see <sup>(1)</sup>), the estimate

$$|u|_s \leq C(K)(|Pu|_{s-m+\delta} + |u|_{s-1}), \quad u \in C_0^\infty(K) \quad (1)$$

( $m$  is the order of the operator  $P$ ,  $K$  is a compact set in  $\Omega \subset \mathbf{R}^n$ ) holds for  $0 \leq \delta < 1/2$  only for elliptic operators, but for  $\delta = 1/2$  this estimate defines a new class of "subelliptic" operators, broader than the class of elliptic operators. In our paper <sup>(3)</sup> it was shown that for  $1/2 \leq \delta < 2/3$  estimate (1) holds only for subelliptic operators. In the same paper examples were given of operators for which estimate (1) holds for  $\delta = k/(k+1)$  (and does not hold for smaller values of  $\delta$ ), and a hypothesis was stated concerning the absence of operators corresponding to intermediate values of  $\delta$ .

In the present paper this hypothesis is proved for the interval  $2/3 < \delta < 3/4$ . Here we make essential use of the exact conditions of an algebraic nature obtained by us, which determine the class of operators for which estimate (1) holds with  $\delta = 2/3$ . These conditions are also of independent interest.

Unlike subelliptic operators, whose characteristic points always form a manifold of dimension  $2n - 2$  in the space of variables  $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ , operators of the indicated class may have a characteristic manifold of any dimension from 1 to  $2n - 2$ . For example, this class includes operators with symbol

$$p(x, \xi) = i\xi_1 + (x_1^2 + \dots + x_k^2)|\xi| + (\xi_1^2 + \dots + \xi_l^2)|\xi|^{-1}$$

$$(1 \leq k \leq n, 1 \leq l \leq n - 1),$$

for which the dimension of the characteristic manifold is equal to  $2n - k - l$ . In addition, the class of operators under consideration contains some scalar differential operators, for example the operator

$$P = x_1 \Delta + ia \partial^2 / \partial x_1^2 \quad (a \neq 0).$$

2. Theorem 1 below gives exact conditions characterizing the operators for which

$$|u|_s \leq C(K)(|Pu|_{s-m+2/3} + |u|_{s-1}), \quad u \in C_0^\infty(K). \quad (2)$$

Let us note the following simple assertion:

**Lemma 1.** *If for an operator  $P$  estimate (1) holds with some  $\delta < 1$ , then at a characteristic point  $(x_0, \xi_0)$  all first-order partial derivatives  $\partial p^0(x_0, \xi_0) / \partial x_k$ ,  $\partial p^0(x_0, \xi_0) / \partial \xi_j$  cannot vanish simultaneously.*

We now give the main result of the present paper.

**Theorem 1.** In order that the estimate (2) hold for the operator  $P$ , it is necessary and sufficient that:

- 1) at the characteristic points

$$\{(x_0, \xi_0), p^0(x_0, \xi_0) = 0, |\xi_0| = 1\}$$

the condition

$$\operatorname{Im} p_x^0(x_0, \xi_0) p_\xi^0(x_0, \xi_0) = \operatorname{Im} \sum_1^n \frac{\partial p^0(x_0, \xi_0)}{\partial x_k} \overline{\frac{\partial p^0(x_0, \xi_0)}{\partial \xi_k}} \geq 0$$

be satisfied;

- 2) if at a characteristic point  $(x_0, \xi_0)$

$$\operatorname{Im} p_x^0(x_0, \xi_0) p_\xi^0(x_0, \xi_0) = 0,$$

then

$$\partial p^0(x_0, \xi_0) / \partial x_j = \kappa \alpha_j, \quad \partial p^0(x_0, \xi_0) / \partial \xi_j = \kappa \beta_j,$$

where  $\alpha_j, \beta_j$  are real numbers,  $j = 1, \dots, n$ ;

- 3) if at a characteristic point  $(x_0, \xi_0)$

$$\operatorname{Im} p_x^0(x_0, \xi_0) p_\xi^0(x_0, \xi_0) = 0,$$

then

$$I_1 = \operatorname{Im} \bar{\kappa} [(Ap'_\xi(x_0, \xi_0), \overline{p'_\xi(x_0, \xi_0)}) - 2(Bp_x^0(x_0, \xi_0), p_\xi^0(x_0, \xi_0)) + (Cp_x^0(x_0, \xi_0), \overline{p_x^0(x_0, \xi_0)})] \neq 0,$$

where

$$A = (\partial^2 p^0(x_0, \xi_0) / (\partial x_i \partial x_j)), \quad B = (\partial^2 p^0(x_0, \xi_0) / \partial x_i \partial \xi_j), \\ C = (\partial^2 p^0(x_0, \xi_0) / \partial \xi_i \partial \xi_j).$$

Moreover,

$$I_1 \operatorname{Im} \kappa[(Ax, x) + 2(B\xi, x) + (C\xi, \xi)] \geq 0$$

for all real  $x$  and  $\xi$  such that

$$(p_x^0(x_0, \xi_0), x) + (p_\xi^0(x_0, \xi_0), \xi) = 0.$$

For the proof of Theorem 1 we use Theorem 1 from our paper (3) and the following lemmas.

**Lemma 2.** For no complex  $a_{ij}$  can one specify such a constant  $C$  that for every  $\psi \in C_0^\infty(\mathbf{R}^n)$  the inequality

$$\int |\psi(y)|^2 dy \leq C \int \left| \frac{\partial \psi}{\partial y_1} + i \frac{\partial \psi}{\partial y_2} + \sum_1^n a_{kj} y_{ky} j \psi \right|^2 dy$$

holds.

**Lemma 3.** There does not exist a constant  $C$ , independent of the parameter  $\lambda$ , for which the inequality

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \leq C \int_{-\infty}^{\infty} |f'(t) + (t^2 - \lambda^2)f(t)|^2 dt, \quad f \in C_0^\infty(\mathbf{R}^1),$$

would hold.

**Lemma 4.** In order that the estimate

$$\begin{aligned} \int |\psi(y)|^2 dy &\leq C_1 \int \left| \frac{\partial \psi}{\partial y_1} + \sum a_{jk} y_{jy} k \psi + 2\lambda^{-1/2} \sum b_{jk} y_{jD} k \psi(y) + \right. \\ &+ \lambda^{-1/3} \left. \sum c_{jk} D_{jD} k \psi(y) \right|^2 dy + C_2 \lambda^{-\varepsilon} \int_{|\alpha+\beta| \leq N} |y^\beta D^\alpha \psi(y)|^2 \lambda^{-2|\alpha|/3} dy, \\ \varepsilon &> 0, \quad \psi \in C_0^\infty(\mathbf{R}^n), \end{aligned}$$

where  $C_1$  and  $C_2$  do not depend on  $\lambda$ , and  $N$  is any number  $\geq 3$ , it is necessary and sufficient that

$$\operatorname{Re} a_{11} \neq 0; \quad \operatorname{Re} a_{11} \cdot \operatorname{Re} \left( \sum_1^n a_{jk} \alpha_j \alpha_k + 2 \sum_{\substack{j=1 \\ k=2}}^n b_{jk} \alpha_j \beta_k + \sum_2^n c_{jk} \beta_j \beta_k \right) \geq 0$$

for all real  $\alpha_1, \dots, \alpha_n, \beta_2, \dots, \beta_n$ .

3. **Theorem 2.** If for the operator  $P$  the estimate

$$|u|_s \leq C(K)(|Pu|_{s-m+\delta} + |u|_{s-1}), \quad u \in C_0^\infty(K),$$

holds, and  $2/3 \leq \delta < 3/4$ , then there exists a constant  $C_1(K)$ , depending on  $K$  but not depending on the function  $u$ , such that

$$|u|_s \leq C_1(K)(|Pu|_{s-m+2/3} + |u|_{s-1}), \quad u \in C_0^\infty(K).$$

For the proof of Theorem 2 one uses not only the result formulated in Theorem 1, but also its proof. In particular, Lemmas 2-4 are used essentially.

4. Analogous theorems are valid for broader classes of operators  $L_{\rho,\delta}^0$  with  $\rho > \delta$ , introduced by L. Hörmander in [4].

As examples of pseudodifferential operators satisfying the conditions of Theorem 1, we mention, in addition to those indicated in item 1, the following:

- 1) the operator with symbol

$$p^0(x, \xi) = (ax_1^2 + bx_2^2)|\xi| - ic\xi_1, \quad ac \neq 0, \quad ab \geq 0,$$

arising in the solution of the problem with oblique derivative

$$\partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2 + \partial^2 u / \partial x_3^2 = 0, \quad x_3 > 0,$$

$$(ax_1^2 + bx_2^2)\partial u / \partial x_3 + c\partial u / \partial x_1 = f(x_1, x_2), \quad x_3 = 0;$$

- 2) the operator with symbol

$$p^0(x, \xi) = ix_1|\xi|^2 + a\xi_1^2 + b\xi_2^2, \quad a > 0, \quad b \geq 0,$$

arising in the solution of the problem:

$$\partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2 + \partial^2 u / \partial x_3^2 + \partial^2 u / \partial x_4^2 = 0, \quad x_4 > 0,$$

$$ix_1\partial^2 u / \partial x_4^2 - a\partial^2 u / \partial x_1^2 - b\partial^2 u / \partial x_2^2 = f(x_1, x_2, x_3), \quad x_4 = 0.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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