# LINEAR POLYNOMIAL OPERATIONS IN THE COMPLEX DOMAIN AND FABER POLYNOMIALS

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## Abstract

## Full Text

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*MATHEMATICS*

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# LINEAR POLYNOMIAL OPERATIONS IN THE COMPLEX DOMAIN AND FABER POLYNOMIALS

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**1°.** Let $C$ be the space of all continuous $2\pi$-periodic functions $f(x)$ with norm

$$\|f\|_{\widetilde{C}} = \max_{0\leq x\leq 2\pi} |f(x)|.$$

Denote by $\Pi_n$ the set of all trigonometric polynomials of order $\leq n$. Let $U_n$ be a linear operation on $C$ into $\widetilde{C}$ having the properties: 1) for every $f \in \widetilde{C}$, $U_n(f) \in \Pi_n$; 2) if $f \in \Pi_n$, then $U_n(f) = f$. Denote the set of all such $U_n$ by $\Omega_{n,n}$. Obviously, the partial sum $S_n(f)$ of the Fourier series of order $n$ of the function $f$ belongs to $\Omega_{n,n}$. The trigonometric interpolation polynomial of Lagrange $L_n(f)$ of order $n$, constructed for an arbitrary system of interpolation nodes, also belongs to $\Omega_{n,n}$. It is known ([1]) that

$$\frac{1}{2\pi}\int_0^{2\pi} (U_n(f_t))_{-t}\, dt = S_n(f), \qquad f \in \widetilde{C}, \quad U_n \in \Omega_{n,n}, \tag{1}$$

where $f_t$ is the shifted function, which is defined according to the equality

$$f_t(x) = f(x+t), \qquad -\infty < t < \infty.$$

Special cases of formula (1), when $U_n(f) = L_n(f)$, are indicated in ([2,3]).

Denote by $A$ the set of all functions $f(z)$, continuous in the disk $K$, $|z| \leq 1$, and analytic at each of its interior points. We define the notion of a shift of a function $f \in A$ by a real number $t$, $-\infty < t < \infty$, by means of the equality

$$f_t(z) = f(ze^{it}). \tag{2}$$

Put

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}\, d\theta, \qquad S_n(f,z) = \sum_{k=0}^{n} c_k z^k, \quad f \in A. \tag{3}$$

By $\Omega_{n,n}(K)$ we denote the set of all linear operations $U_n$ from $A$ into $A$ having the properties: 1) for any $f \in A$, $U_n(f)$ is a polynomial of degree $\leq n$; 2) if $f$

is a polynomial of degree $\leq n$, then $U_n(f) = f$. In ([4]) it is proved that for the disk $K$ there is an analogue of formula (1), namely

$$\frac{1}{2\pi}\int_0^{2\pi}(U_n(f_t))_{-t}\,dt = S_n(f), \tag{4}$$

where $S_n(f)$ is defined according to equality (3), and $f_t$ according to equality (2).

**2°.** Let $D$ be an arbitrary finite domain with simply connected complement $D_1$ and rectifiable boundary $\Gamma$. By $w = \Phi(z)$ we denote the function mapping $D_1$ conformally onto the domain $|w| \geq 1$ of the $w$-plane under the condition $\Phi(\infty) = \infty$. Let $z = \Psi(w)$ be the inverse function. By $A(D)$ we denote the set of all functions $f(z)$, continuous in the closed domain $\overline{D}$ and analytic at each interior point of $D$. Put $\|f\| =$

$$= \max_{z\in\overline{D}}|f(z)|.$$

Define the translate $f_t$ of a function $f \in A(D)$ by a real number $t$, $-\infty < t < \infty$, by means of the equality

$$f_t(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f\{\Psi[\Phi(\xi)e^{it}]\}}{\xi - z}\,d\xi. \tag{5}$$

It is not hard to see that if the domain $D$ is a disk $K$, then the translate defined by formula (5) coincides with the translate defined according to (2). Denote by $F_k(z)$ the Faber polynomial of degree $k$ generated by the domain $D$ ([5–7]). It is curious that the polynomials $F_k$ are transformed very simply under the translate (5). Namely, it is easy to prove that

$$(F_k)_t = F_k e^{ikt}. \tag{6}$$

This equality is a special case of an identity of V. K. Dzyadyk ([8]).

Let $U_n$ be a linear operation from $A(D)$ into $A(D)$ possessing the properties: 1) for any $f \in A$, $U_n(f)$ is a polynomial of degree $\leq n$; 2) if $f$ is a polynomial of degree $\leq n$, then $U_n(f) = f$. The set of all such $U_n$ will be denoted by $\Omega_{n,n}(D)$. The most important example of an operation of type $U_n$ is furnished by the partial sum

$$S_n(f) = S_n(f,z) = \sum_{k=0}^{n} a_k F_k(z), \qquad a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f[\Psi(e^{it})]e^{-ikt}\,dt, \tag{7}$$

of the expansion of a function $f \in A(D)$ in Faber polynomials.

It turns out that there is a very simple relation between an arbitrary operation $U_n \in \Omega_{n,n}(D)$ and the operation (7).

**Theorem 1.** For any $f \in A(D)$ and any $U_n \in \Omega_{n,n}(D)$, the equality

$$\frac{1}{2\pi}\int_0^{2\pi} (U_n(f_t))_{-t}\,dt = S_n(f)$$

holds, where $S_n(f)$ is defined according to (7) and $f_t$ according to (5).

**Proof.** Put

$$\widetilde{U}_n(f) = \frac{1}{2\pi}\int_0^{2\pi} (U_n(f_t))_{-t}\,dt. \tag{8}$$

Then it is necessary to prove that

$$\widetilde{U}_n(f) = S_n(f). \tag{9}$$

Since the Faber polynomials form a basis in $A(D)$, it is enough to prove that (9) holds when $f$ is a Faber polynomial. Suppose first that $f = F_k$, where $k \le n$. Then, by virtue of (6) and the definition of $U_n$, we have

$$[U_n((F_k)_t)]_{-t} = F_k e^{ikt} e^{-ikt} = F_k.$$

Consequently, $\widetilde{U}_n(F_k) = F_k$. On the other hand, $S_n(F_k) = F_k$. Therefore $\widetilde{U}_n(F_k) = S_n(F_k)$. Let now $f = F_k$, where $k > n$. By virtue of (6),

$$U_n((F_k)_t) = e^{ikt} U_n(F_k). \tag{10}$$

Since $U_n \in \Omega_{n,n}(D)$, we have $U_n(F_k) = \sum_{j=0}^{n} a_j F_j$, where $\{a_j\}$ are certain numbers. Therefore, according to (6) and (10),

$$[U_n((F_k)_t)]_{-t} = e^{ikt} \sum_{j=0}^{n} a_j F_j e^{-ijt}. \tag{11}$$

It follows from (11) that $\widetilde{U}_n(F_k) = 0$ for $k > n$. Since $S_n(F_k) = 0$, $k > n$, (9) again holds. Thus the theorem is proved.

Obviously, equality (4) is a particular case of Theorem 1, when the domain $D$ is a disk.

**Corollary.** Let $\{z_k^{(n+1)}\}_{k=1}^{n+1}$ be points from $\overline{D}$, with $z_k^{(n+1)} \neq z_l^{(n+1)}$, $k \neq l$. Denote by $L_n(f,z)$ the Lagrange interpolation polynomial of degree $n$, constructed for $f \in A(D)$ at the nodes $\{z_i^{(n+1)}\}_{i=1}^{n+1}$. Then the identity holds

$$\frac{1}{2\pi}\int_0^{2\pi} (L_n(f_t))_{-t}\,dt = S_n(f),$$

where $S_n(f)$ is defined according to equality (7).

Introduce the operator

$$\sigma_n(f) = \sigma_n(f,z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{-t}(z)K(t)\,dt, \tag{12}$$

where $K \in \Pi_n$ and $f_{-t}$ is defined according to (5). Operators of this kind were considered in (8).

**Theorem 2.** Let $U_n$ be a linear operation from $A(D)$ into $A(D)$ having the following properties: 1) for every $f \in A(D)$, $U_n(f)$ is a polynomial of degree $\leq n$; 2) if $f$ is a polynomial of degree $\leq n$, then $U_n(f) = \sigma_n(f)$, where $\sigma_n(f)$ is defined according to (12). Then

$$\frac{1}{2\pi}\int_0^{2\pi} (U_n(f_t))_{-t}\,dt = \sigma_n(f).$$

Obviously, Theorem 1 is a particular case of Theorem 2.

3°. We shall say that an operator $C$ **commutes with the shift** if $C(f_t) = (C(f))_t$, $-\infty < t < \infty$, where the shift is defined according to (5). It is not difficult to verify that the operator (12) commutes with the shift. It turns out that among all operators satisfying the conditions of Theorem 2 and commuting with the shift, it is the only one, for the following holds:

**Theorem 3.** Let $U_n$ satisfy the conditions of Theorem 2 and commute with the shift; then for every $f \in A(D)$,

$$U_n(f) = \sigma_n(f).$$

This theorem follows from Theorem 2 and the equality $(f_t)_{-t} = f$.

Denote by $\Omega_n(D)$ the set of all linear operations from $A(D)$ into $A(D)$ having the property that, for every $f \in A(D)$, $U_n(f)$ is a polynomial of degree $\leq n$. Obviously, $\Omega_{n,n}(D) \subset \Omega_n(D)$. For operators from $\Omega_n(D)$ the following theorem is valid:

**Theorem 4.** Let $\widetilde{U}_n \in \Omega_n(D)$. Then for every $f \in A(D)$ the equality holds

$$\widetilde{U}_n(f) = \widetilde{U}_n(S_n(f)), \tag{13}$$

where $\widetilde{U}_n$ is defined according to (8), and $S_n(f)$ is the partial sum of the expansion of the function $f$ in Faber polynomials.

**Proof.** Since the Faber polynomials form a basis in $A(D)$, it is sufficient to prove identity (13) only for Faber polynomials. Let $f = F_k$, $k \leq n$. Then $S_n(F_k) = F_k$. Therefore $\widetilde{U}_n(S_n(F_k)) = \widetilde{U}_n(F_k)$. Thus (13) holds. Now let $f = F_k$, where $k > n$. Then $S_n(F_k) = 0$. Consequently, $\widetilde{U}_n(S_n(F_k)) = 0$. On the other hand, according to equality (11), which is also valid for operators of the class $\Omega_n(D)$, we have $\widetilde{U}_n(F_k) = 0$. Thus, for $k > n$, (13) also holds.

Let $U_n^{(i)} \in \Omega_n(D)$, $i = 1, 2$. Suppose that for every polynomial $f$ of degree $\leq n$, $U_n^{(1)}(f) = U_n^{(2)}(f)$. It does not at all follow from this equality that for every $f \in A(D)$, $U_n^{(1)}(f) = U_n^{(2)}(f)$. However, the following is valid:

**Theorem 5.** If the operators $U_n^{\langle 1\rangle}$ and $U_n^{\langle 2\rangle}$ from the class $\Omega_n(D)$ coincide on the set of all polynomials of degree $\leqslant n$, then, for any $f \in A(D)$,

$$\widetilde{U}_1(f) = \widetilde{U}_2(f).$$

**Proof.** Since $S_n(f)$ is a polynomial of degree $\leqslant n$, by the hypothesis of the theorem,

$$U_n^{\langle 1\rangle}(S_n(f)) = U_n^{\langle 2\rangle}(S_n(f)).$$

Therefore, by Theorem 4,

$$\widetilde{U}_n^{\langle 1\rangle}(f) = \widetilde{U}_n^{\langle 2\rangle}(f)$$

for every $f \in A(D)$.

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## CITED LITERATURE

1. D. L. Berlin, DAN, 85, No. 1 (1952).

2. G. Faber, Jahresber. DMV, 23, 192 (1914).

3. J. Marcinkiewicz, Acta Szeged, 8, 127 (1937).

4. D. L. Berman, Izv. vyssh. uchebn. zaved., matem., 3 (52), 24 (1966).

5. G. Faber, Math. Ann., 57, 389 (1903).

6. A. I. Markushevich, *Theory of Analytic Functions*, Moscow-Leningrad, 1950.

7. V. I. Smirnov, N. A. Lebedev, *Constructive Theory of Functions of a Complex Variable*, 1964.

8. V. K. Dzyadyk, Izv. AN SSSR, ser. matem., 26, 797 (1962).

*Note: Figure translations are in progress. See original paper for figures.*