ON THE APPROXIMATION VISCOSITY OF DIFFERENCE SCHEMES

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Abstract

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ON THE APPROXIMATION VISCOSITY OF DIFFERENCE SCHEMES

1. In the integration of hyperbolic systems by finite-difference methods, effects arise of so-called approximation viscosity, connected with the dissipative character of difference schemes. In a certain sense, the approximation viscosity of a scheme and its dissipative properties are determined by the first differential approximation, the concept of which is given below. Two notions will be given of schemes possessing approximation viscosity in the sense of preservation (non-smearing) of contact discontinuities, and for a number of schemes conditions will be established under which they possess this property.

2. Let the hyperbolic system

$$\partial u/\partial t = A\,\partial u/\partial x \tag{1}$$

be approximated by the difference scheme

$$u^{n+1}(x) = \sum_\alpha B_\alpha u^n(x+\alpha h) = \Omega(T)u^n(x). \tag{2}$$

Here A is a constant real $m \times m$ matrix with distinct real eigenvalues; $u(x,t) = \{u_1(x,t), \dots, u_m(x,t)\}$. The B_α are constant real $m \times m$ matrices; T is the shift operator with respect to x; $t = n\tau$.

In view of the approximation, the following consistency conditions are satisfied:

$$\sum_\alpha B_\alpha = I, \qquad \sum_\alpha \alpha B_\alpha = \frac{\tau}{h}A.$$

I is the identity matrix.

Expanding in (2) the functions $u^{n+1}(x)$, $u^n(x+\alpha h)$ in series in the parameters τ and h about the point (x,t), and discarding terms above first order in τ and h, we have

$$\partial u/\partial t = A\,\partial u/\partial x + C\,\partial^2 u/\partial x^2,$$

$$C = \frac{h^2}{4\tau}\sum_{\alpha,\beta}(\alpha-\beta)^2 B_\alpha B_\beta. \tag{3}$$

We shall call system (3) the first differential approximation of the difference scheme (2). Here and below the first differential approximation is considered on solutions.

3. The harmonic $u = u_0 e^{\omega t + ikx}$ is a solution of (1) if ω and k are related by the dispersion relation

$$\det\|\omega I - ikA\| = 0. \tag{4}$$

If for some family $\omega = \omega(k)$ of solutions of (4) $\omega(k) \equiv 0$, then $Au_0 = 0$, and the corresponding invariant of system (1) is transported along the characteristic without change (the contact boundary is preserved).

Definition 1. The difference scheme (2) possesses approximation viscosity preserving the contact discontinuity (**property** K) if from the relation $Au_0 = 0$ it follows that $Cu_0 = 0$.

Lemma 1. If from $Au_0 = 0$ it follows that

$$\sum_\alpha \alpha^2 B_\alpha u_0 = 0,$$

then scheme (2) has property K.

Theorem 1. If:

1) $C = F \cdot A$

or

2)

$$\sum_\alpha \alpha^2 B_\alpha = H \cdot A,$$

or

3) $B_\alpha = F_\alpha \cdot A$,

where F, H, F_α are certain matrix functions, then scheme (2) has property K.

Theorem 2. In order that the simple difference scheme

$$u^{n+1}(x) = \sum_{\alpha=1}^{2} B_\alpha u^n(x+\tau\lambda_\alpha), \qquad \sum_{\alpha=1}^{2} B_\alpha = I, \qquad \sum_{\alpha=1}^{2} \lambda_\alpha B_\alpha = A \tag{5}$$

have property K, it is necessary and sufficient that one of the following conditions be satisfied:

1) $\lambda_1 = 0 \quad (\lambda_2 \neq 0)$

or

2) $\lambda_2 = 0 \quad (\lambda_1 \neq 0)$.

Theorem 3. The majorant scheme

$$u^{n+1}(x) = \sum_{\alpha=-1}^{1} B_\alpha u^n(x + \alpha h),$$

$$B_1 = \frac{\tau}{h} A^+, \qquad B_{-1} = -\frac{\tau}{h} A^-, \qquad B_0 = I - (B_1 + B_{-1}), \tag{6}$$

$$A = A^+ + A^-, \qquad A^+ \geqslant 0, \qquad A^- \leqslant 0$$

has property K.

Consider the predictor-corrector scheme:

$$u^*(x) = \sum_\alpha B_\alpha u^n(x + \tau^* \lambda_\alpha), \tag{7}$$

$$u^{n+1}(x) = u^n(x) + \frac{\tau}{h} A\,[u^*(x + h/2) - u^*(x - h/2)], \tag{8}$$

where (7) is an arbitrary difference scheme, and (8) is the "cross" scheme.

Theorem 4. The predictor-corrector scheme (7), (8) has property K.

4. **Definition 2.** Difference scheme (2) has the **strong property** K if from the relation $Au_0 = 0$ it follows that

$$[\Omega(e^{ikh}) - I]u_0 = 0.$$

Here

$$\Omega(e^{ikh}) = \sum_\alpha B_\alpha e^{i\alpha kh}$$

is the amplification matrix of scheme (2).

Theorem 5. If $\Omega(e^{ikh}) - I = F \cdot A$, where F is an arbitrary matrix function, then scheme (2) has the strong property K.

Theorem 6. In order that the simple scheme (5) have the strong property K, it is necessary and sufficient that one of the following conditions be satisfied: 1) $\lambda_1 = 0$ ($\lambda_2 \neq 0$) or 2) $\lambda_2 = 0$ ($\lambda_1 = 0$).

Theorem 7. The majorant scheme (6) has the strong property K.

Theorem 8. If $B_\alpha A = AB_\alpha$, then the predictor-corrector scheme (7), (8) has the strong property K.

Corollary 1. The Lax scheme (the simple scheme with $\lambda_1 = h/\tau$, $\lambda_2 = -h/\tau$) does not have property K in the general case.

Corollary 2. The Lax–Wendroff scheme has property K in the general case.

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Note: Figure translations are in progress. See original paper for figures.