GROUP PROPERTIES (IN THE LARGE) OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

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GROUP PROPERTIES (IN THE LARGE) OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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1. The general investigation of the group properties of differential equations was initiated by S. Lie ([1]). L. V. Ovsyannikov ([2]) proposed a constructive method for finding the Lie group of the linear homogeneous second-order partial differential equation

$$F(u) \equiv a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i(x)\frac{\partial u}{\partial x^i} + c(x)u = 0, \tag{1}$$

where $x = (x^1, x^2, \dots, x^n)$, $n \geq 2$. Like the whole of Lie theory, L. V. Ovsyannikov' s method is a local method, making it possible to study only infinitesimal transformations of the group.

In ([3]) a connection was established between the generalized Schrödinger equation and a Riemannian space, which made it possible to describe the symmetry group of the Schrödinger equation in the large. The main point of ([3]) is the writing of the Schrödinger equation in the form of an equation corresponding to the free motion of a particle in a certain Riemannian space V_n,

$$\Delta_2 u + \frac{n-2}{4(n-1)}Ru = \pm u; \tag{2}$$

Δ_2 and R are, respectively, the Laplace-Beltrami operator and the scalar curvature of V_n.* Equation (2), which is explicitly invariant with respect to the group of motions of the space V_n, will be called the canonical Schrödinger equation. In ([4]) all equations equivalent to (2) were found. We note that reducing an equation to canonical form is useful in the solution of concrete problems ([5],[6]).

In the present paper the symmetry groups of equation (1) are investigated. By the symmetry group of an equation is meant the totality of all sufficiently smooth transformations of the dependent and independent variables that preserve the left-hand side of (1) up to some factor (since finite transformations are involved, we speak of the group of equation (1) in the large). The method used is based on considering the set of equivalence transformations of equation (1), an element of which is the transformation of (1) to the canonical form, which is a generalization of (2). In conclusion, conditions necessary and sufficient for separation of variables in equation (1) are established.

2. Under the assumption of nondegeneracy of the tensor a^{ij}, we have

$$F(u) \equiv \Delta_2 u + a^i \partial u/\partial x^i + cu = 0, \tag{1'}$$

where $\Delta_2 \equiv a^{ij}(\partial^2/\partial x^i \partial x^j - \Gamma^k_{ij}\partial/\partial x^k)$ is the Laplace–Beltrami operator ([7]) of the Riemannian space V_n with metric $ds^2 = a_{ij}dx^i dx^j$ ($a_{jk}a^{ji} = \delta^i_k$; $i, k = 1, 2, \ldots, n$; Γ^k_{ij} are Christoffel symbols); $a^i = b^i + a^{kl}\Gamma^i_{kl}$. Con-

* Thus, for example, the ordinary Schrödinger equation in three-dimensional Euclidean space

$$\Delta_2 u + (E - V(x))u = 0$$

is equivalent to equation (2) in the space with metric

$$ds^2 = (E - V)(dx^2 + dy^2 + dz^2).$$

consider equations equivalent to (1)

$$\bar{F}(u) \equiv \exp[-\theta(x) - \nu(x)]F(\exp \nu(x) \cdot u) \equiv \bar{\Delta}_2 u + +\bar{a}^i \partial u/\partial x^i + \bar{c}u = 0. \tag{3}$$

Here $\theta(x)$ and $\nu(x)$ are sufficiently smooth functions; the operator $\bar{\Delta}_2$ is defined on the metric $d\bar{s}^2 = \bar{a}_{ij}dx^i dx^j = e^\theta a_{ij}dx^i dx^j$ of the space $\bar{V}_n$, conformal to V_n;

$$\bar{a}^i = e^{-\theta}\left[a^i + a^{ij}\partial\left(2\nu + \frac{2-n}{2}\theta\right)/\partial x^j\right]$$

(((4), formula (7)), i.e.

$$\bar{a}_i = \bar{a}_{ij}\bar{a}^j = a_i + \partial\left(2\nu + \frac{2-n}{2}\theta\right)/\partial x^i. \tag{4}$$

Substituting into $\bar{c} = e^{-\theta}(c + b^i\nu_i + a^{ij}\nu_i\nu_j + a^{ij}\nu_{ij})$ the quantities ν_i from (4) (here $\nu_i = \partial\nu/\partial x^i$, $\nu_{ij} = \partial^2\nu/\partial x^i \partial x^j$) and using formula (28.7) of work (7), we obtain

$$\bar{F}(u) \equiv \exp[-\theta(x) - \nu(x)]F(\exp \nu(x) \cdot u) = \bar{\Delta}_2 u + \bar{a}^i \partial u/\partial x^i +$$

$$+\left[\frac{n-2}{4(n-1)}\bar{R} + \frac{1}{4}\bar{a}^i\bar{a}_i + \frac{1}{2}\bar{a}^i{}_{,i}\right]u +$$

$$+e^{-\theta}\left[c-\frac{n-2}{4(n-1)}R-\frac{1}{4}a^i a_i-\frac{1}{2}a^i{}_{,i}\right]u=0, \tag{5}$$

where the scalar curvature $R(\bar{R})$ and the covariant derivatives $a^i{}_{,j}(\bar{a}^i{}_{,j})$ (7) are defined in accordance with the metric $ds^2 = a_{ij}dx^i dx^j$ ($d\bar{s}^2 = e^\theta a_{ij}dx^i dx^j$). If the function $H = c - (n-2)R/4(n-1) - {}^1/4\, a^i a_i - {}^1/2\, a^i{}_{,i}$ does not vanish, then $\theta = \ln|H|$ leads to

$$\bar{\Delta}_2 u + \bar{a}^i\frac{\partial u}{\partial x^i} + \left[\frac{n-2}{4(n-1)}\bar{R} + \frac{1}{4}\bar{a}^i\bar{a}_i + \frac{1}{2}\bar{a}^i{}_{,i}\right]u = \pm u, \tag{6}$$

where the sign in the right-hand side is determined as $-\operatorname{sign} H$ (for $H < 0$ both parts of equation (5) must be multiplied by -1). We shall call the form (6) the canonical form of equation (1).

Thus, if for equation (1) $H \neq 0$, then equation (1) is equivalent to the form (6), determined with the aid of the tensor Ha_{ij} and the vector a_i (the latter is determined up to the gradient of an arbitrary function).

From (4) it follows that if the field $a_i(x)$ is potential, i.e.

$$\partial a_i/\partial x^j - \partial a_j/\partial x^i = 0 \quad (i, j = 1, 2, \dots, n), \tag{7}$$

then the system of equations $\bar{a}_i = 0$ is solvable, and the transformation (5) with

$$\nu(x) = \frac{n-2}{4}\theta(x) - \frac{1}{2}\int a_i dx^i$$

leads to an equation for which $\bar{a}_i \equiv 0$; in this case the form (6) coincides with (2) ((4), theorem 2*). The case $H \equiv 0$ corresponds, obviously, to the form (6) with right-hand side equal to zero, determined on the metric $d\bar{s}^2 = e^{\theta(x)}ds^2$.

3. By the symmetry group G of equation (1) we shall understand the set of sufficiently smooth transformations of the dependent and independent variables $x' = x'(x)$ and $u' = \eta(x, u)$, under which the equation preserves its form up to a certain factor $\lambda(x) \neq 0$, i.e. $F'(x', u') = \lambda F(x', u')$**. To preserve the linearity and homogeneity of equation (1), it is necessary that $\eta(x, u) = \sigma(x)u + \varphi(x)$, where $F(\varphi) = 0$. The group G contains the subgroup T of substitutions $x' = x,\ u' = tu + \varphi$ (t is a numerical parameter, φ is any solution of (1)), conditioned by the linearity of (1). In what follows, by the group of equation (1) we mean the transformations of the factor group G/T: $x' = x'(x),\ u' = \sigma(x)u$.

* The possibility of generalizing theorem 2 of work (4) to the case of an arbitrary vector field was noted by A. S. Shvartsem (private communication).

** Consideration of transformations of the form $x' = x'(x, u),\ u' = \eta(x, w),\ \lambda = \lambda(x, u, \partial u/\partial x^i)$ does not lead to an enlargement of the group G (2).

Theorem 1. The group G/T of equation (1) is isomorphic to a subgroup of the group of conformal transformations $x' = x'(x)$ of a Riemannian space with metric $ds^2 = a_{ij}dx^i dx^j$, for which $a'_{ij}(x')H(x(x')) = a_{ij}(x')H(x')$, while the vector $a_i(x)$ is preserved up to the gradient of an arbitrary function, i.e.

$$a'_i(x') = a_i(x') + \partial\left(2\nu + \frac{2-n}{2}\theta\right)/\partial x'^i.$$

The definition of invariance of the equation used by us is, evidently, equivalent to the requirement that (1), in the new variables x', u', coincide with one of equations (5). Under the change $x' = x'(x)$ (it is assumed that the Jacobian of the transformation is nonzero), equation (1), written in the form (5) for $\theta = \nu = 0$, will be

$$\Delta'_2 u + a'^i\frac{\partial u}{\partial x'^i} + \left[\frac{n-2}{4(n-1)}R' + \frac{1}{4}a'^i a'_i + \frac{1}{2}a'^i_{,i}\right]u + H'(x')u = 0. \tag{8}$$

Here Δ'_2 and R' are defined by the metric ${ds'}^2 = a'_{ij}(x')dx'^i dx'^j$ of the space V_n in the new variables x'. Comparing the coefficients of the second derivatives in (8) and (5), we obtain $a'^{ij}(x') = e^{-\theta(x')}a^{ij}(x')$. Thus the transformation $x' = x'(x)$ must be conformal,

$${ds'}^2 = e^{\theta(x')}a_{ij}(x')dx'^i dx'^j, \tag{9}$$

and equation (8) in the system (x') of the space V_n may be regarded as equation (5) on the metric $ds^2 = e^{\theta(x)}a_{ij}dx^i dx^j$ of the space $\bar{V}_n$, conformal to V_n (it is enough to replace x by x' in (5)). Comparison of the coefficients of equations (8) and (5) further gives $H(x(x')) = e^{-\theta(x')}H(x')$,

$$a'_i(x') = a_i(x') + \partial\left(2\nu + \frac{2-n}{2}\theta\right)/\partial x'^i \quad (i = 1, 2, \dots, n), \tag{10}$$

which, together with (9), proves theorem 1.

If $H \neq 0$, equation (1) can be reduced to the form (6), invariant with respect to the subgroup G_0 of the group of isometries of the space $\bar{V}_n$ with metric tensor $H(x)a_{ij}(x)$. G_0 consists of transformations satisfying (10) for $\theta = \ln|H(x')|$. On the other hand, every transformation satisfying theorem 1 belongs to G_0.

Theorem 2. If for equation (1) the quantity $H \neq 0$, then the group G/T admitted by this equation is a subgroup of the group of isometries of a Riemannian space with metric $ds^2 = Ha_{ij}dx^i dx^j$.

If (7) is satisfied, (1) is reduced to the form (2), manifestly invariant with respect to the group of isometries of the space with metric $ds^2 = Ha_{ij}dx^i dx^j$ ([3], [4]).

Let $H = 0$. Then it follows from (5) that G/T is a subgroup of the group of conformal transformations of V_n with metric $ds^2 = a_{ij}dx^i dx^j$, satisfying (10). If, moreover, the conditions (7) are satisfied, then G/T is the group of all conformal transformations of V_n.

4. We shall say that equation (1) admits, in the coordinate system $(x^1, x^2, \dots, x^n)$, P-separation of variables if it has solutions of the form

$$u = P^{-1}(x)\prod_{i=1}^{n} u_i(x^i),$$

where $P(x) \neq 0$ is a sufficiently smooth, completely determined function, and $u_i(x^i)$ is any solution of an ordinary differential equation of the second order in x^i $(i = 1, 2, \dots, n)$. Then equation (5), for $e^{-\nu} = |P|$ and $\theta \equiv 0$, admits complete separation in the system $(x^1, x^2, \dots, x^n)$ ([8]). Equation (5) (as also (1′)) may be regarded as a generalized Schrödinger equation in V_n with vector potential $\bar{a}^i(x)$ and scalar potential $V(x)$ ($\bar{c}(x) = E - V(x)$, E is the energy). The presence of the vector field $\bar{a}^i(x)$ leads to the following change in the known conditions ([8], [9]), necessary and sufficient for separation of variables in equation (1′) in the case $\bar{a}^i \equiv 0$.

Theorem 3. For P-separation of variables in equation (1) in the coordinate system $(x^1, x^2, \dots, x^n)$, it is necessary and sufficient that there exist a sufficiently smooth function $P(x) \neq 0$, n^2 functions $\varphi_{ij} = \varphi_{ij}(x^i)$, and n functions $\chi_i = \chi_i(x^i)$ such that, in the system $(x^1, x^2, \dots, x^n)$, the tensor $a^{ij}(x)$, the vector $a^i(x)$, and the scalar $c(x)$ satisfy the conditions:

$$a^{ii} = (\varphi^{-1})_{i1}, \qquad a^{ij} = 0 \quad (i, j = 1, 2, \dots, n;\ i \neq j); \tag{11}$$

$$\partial a_i/\partial x^j = \partial^2 \ln P^2/\partial x^i \partial x^j - \frac{2}{3}R_{ij} \quad (i \neq j); \tag{12}$$

$$c = \sum_{i=1}^{n} \frac{P}{\sqrt{a}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{a}}{P^2 a_{ii}} \frac{\partial P}{\partial x^i}\right) + (\varphi^{-1})_{i1}\chi_i; \tag{13}$$

where $\left|(\varphi^{-1})_{ij}\right|$ is the determinant reciprocal to the Stäckel determinant $\left|\varphi_{ij}(x^i)\right|$, $a = \det|a_{ij}|$, and R_{ij} is the Ricci tensor (7), defined in accordance with the metric $ds^2 = a_{ij}dx^i dx^j$.

Conditions (11) and (13) coincide with the corresponding conditions for the Schrödinger equation with scalar potential [(8)] and are proved analogously. The sufficiency of (12) follows from writing (5) (for $\theta = 0,\ e^{-v} = |P|$) in the form $F(u) = (\varphi^{-1})_{i1} F_i(u) = 0$, where the equations

$$F_i(u) \equiv \frac{d^2u}{dx^{i2}} + \left(\frac{\partial}{\partial x^i} \ln \frac{\sqrt{a}}{a_{ii}} + \bar{a}_i\right) \frac{du}{dx^i} + (a_j \varphi_{ij} + \chi_i) u = 0 \tag{14}$$

will, when (12) is satisfied, be separated (for a metric satisfying (11), we have [8] $R_{ij} = \dfrac{3\partial^2}{2\partial x^i \partial x^j} \ln \dfrac{\sqrt{a}}{\varphi}$, $\varphi = \det|\varphi_{ij}|$, $\alpha_i = \text{const}$). The proof of the necessity of (12) is carried out as in [9].

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Note: Figure translations are in progress. See original paper for figures.