ON INVERSE APPROXIMATION THEOREMS ON CLOSED SETS OF THE COMPLEX PLANE

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.537

MATHEMATICS

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(Presented by Academician V. I. Smirnov on 1 VI 1967)

In the work (1) V. K. Dzyadyk obtained inverse theorems on the approximation of functions by polynomials on continua with connected complement, the boundary of which consists of a finite number of arcs with continuous curvature (under certain additional restrictions, some of which have been removed in recent works of L. I. Kolesnik (2) and N. N. Vorob' eva (3)). N. A. Lebedev (4,5) obtained inverse approximation theorems by rational functions on sets of a more general nature. The present note is devoted to certain generalizations of his results.

We introduce the notation needed below. Let $\overline{B}$ be a bounded closed set whose complement in the extended plane consists of s domains B^j, $j = 1, \dots, s$, pairwise having no common points; L be the boundary of $\overline{B}$; B^j, $j = s+1, s+2, \dots$, be supplementary domains for L belonging to $\overline{B}$ (there are at most a countable number of them); a_j be a fixed point of B^j, $j = 1, \dots, s$. By $R_n(z)$ we shall denote rational functions (r.f.) of the form

$$R_n(z) = \sum_{j=1}^{s} \sum_{\nu=1}^{n} \frac{A_{j,\nu}}{(z-a_j)^{\nu}} + A \tag{1}$$

(if $a_{j_0} = \infty$, then $(z - a_{j_0})^{-\nu}$ in (1) must be replaced by z^{ν}).

Let $\varphi(z)$, $z \in \partial B^j$, $j = 1, 2, \dots$, be a continuous real function; $H_j(\zeta, \varphi(z))$ be the solution of the Dirichlet problem for B^j and $\varphi(z)$ (see, for example, (6)); $g_j(\zeta; a_j) = H_j(\zeta, \ln|z - a_j|) - \ln|\zeta - a_j|$, $j = 1, \dots, s$ (if $a_{j_0} = \infty$, then $g_{j_0}(\zeta, a_{j_0}) = H_{j_0}(\zeta, \ln 1/|z|) - \ln 1/|\zeta|$); $L^j_\rho = \{z : g_j(z, a_j) = \ln \rho\}$, $\rho > 1$; $L_\rho = \bigcup_{j=1}^{s} L^j_\rho$; $d(z, u)$, $u > 0$, be the distance from the point z to L_{1+u}. We further assume that the set $\overline{B}$ has the property that for every $z \in L$, $d(z, u) \to 0$ as $u \to 0$.

Every continuous nondecreasing positive function $\omega(x)$, $x > 0$, $\omega(+0) = 0$, will be called a modulus of continuity (m.c.). Put $G^{\omega}(\zeta, u) = H_j(\zeta, \ln \omega(d(z, u)))$ for $\zeta \in B^j$, $j = 1, 2, ...$, and $G^{\omega}(\zeta, u) = \ln \omega(d(\zeta, u))$ for $\zeta \in L$. If there exists $M > 1$ such that for all $z_0 \in L$, $u > 0$, and ζ_0, related by the condition $|\zeta_0 - z_0| = d(z_0, u)$, the inequality $I \equiv G^{\omega}(\zeta_0, u) - G^{\omega}(z_0, u) \le \ln M$ holds, then we write $\omega(x) \in \{\overline{B}, M\}$.

Next we consider m.c. $\omega(x)$ for which there exists a constant K such that for all $x > 0$

$$K \int_{e^{-1}x}^{x} \omega(t) t^{-2}\, dt \ge \omega(x) x^{-1}, \tag{2}$$

and we denote

$$K_{\omega}^{(p)} = \inf_{x>0} \left(\int_{e^{-1}}^{1} \frac{\omega(\tau x)}{\omega(x)} \tau^{p-1}\, d\tau \right)^{-1}, \qquad p \ge -1.$$

(In this definition, instead of e^{-1} one may put any a, $0 < a < 1$.)

Changing the proof of the theorem from (5) only slightly, we obtain the following generalization of it.

Theorem. Let: $f(z)$ be a function given on the boundary L of the set $\overline{B}$; λ, $\lambda > 0$, be a fixed number; r be a nonnegative integer; $\omega(x)$ be a m.c. satisfying condition (2), and $x^r \omega(x) \in \{\overline{B}, M\}$; $R_m(z)$, $R_{m+1}(z)$, ... be a sequence of r.f. of the form (1) such that

$$|f(z) - R_n(z)| \le d(z, \lambda/n)^r \omega(d(z, \lambda/n)), \qquad z \in L. \tag{3}$$

Denote by $\omega(x, \varphi)$ the modulus of continuity of the function $\varphi(z)$, $z \in L$. Then:

1) if $r = 0$ and $0 < x < e^{-1} \inf_{z \in L} d(z, \lambda/m) \equiv x_0$, we have

$$\omega(x, f) \le \omega(x, R_m) + C_0 M x \int_{x}^{l_m} \omega(t) t^{-2}\, dt,$$

where $l_m = \sup_{z \in L} d(z, \lambda/m)$ and C_0 is a constant

$$\left(0 < C_0 < 4,2 e^{\lambda_0} K_{\omega}^{(-1)}\right),$$

$$\lambda_0 = \max\left\{\frac{m+1}{m}\lambda, \lambda + 2\right\};$$

2) if $r > 0$, $\nu = 1, 2, \ldots, r$,

$$|f^{(\nu)}(z) - R_n^{(\nu)}(z)| \le C_r^{(\nu)} M\Omega_{r-\nu}(d(z,\lambda/n)), \quad z\in L, \quad n=m,m+1,\dots,$$

$$\Omega_p(x)=\int_0^x \omega(t)t^{p-1}\,dt \qquad \left(0<C_r^{(\nu)}<4,2\nu! e^{\lambda_0}K_\omega^{(\nu-r)}\right)$$

(for $\nu = r$ in the case where $\Omega_0(x) < \infty$).

Remark 1. In the statement of the theorem one may take M equal to the exact lower bound of the numbers M' such that $x^r\omega(x) \in \{\overline{B}, M'\}$.

Remark 2. One can obtain an analogous theorem with local m.c., for example, by adding to the right-hand side of inequality (3) a positive factor $\rho(z)$.

Remark 3. By making the proof somewhat more complicated, one can obtain a theorem for sets $\overline{B}$ whose complement consists of an infinite number of domains (under certain additional restrictions on $\overline{B}$).

Remark 4. The theorem can be extended to unbounded sets. For this it is necessary to pass to the metric defined by the fact that the distance between any two points in the plane is understood as the distance between the corresponding points on the Riemann sphere (this is natural in the study of approximation by r.f.).

The question arises: for every m.c. $\omega(x)$ satisfying condition (2), with fixed $\overline{B}$ and a_j, $j = 1, \dots, s$, does there exist $M > 1$ such that $\omega(x) \in \{\overline{B}, M\}$? At the International Congress of Mathematicians in 1966 N. A. Lebedev, in his communication, expressed the conjecture that for the m.c. $\omega(x)$ considered in (5), the answer to this question is affirmative. There, at the congress, this conjecture was proved by P. M. Tamrazov. Below, by the same method, a more general assertion is proved.

Let the diameters of the continua composing the set $\overline{B}$ be bounded below by some number $\beta > 0$. We shall call such a set a set of type $\mathfrak{M}_s^\beta$.

Lemma 1. Let $\overline{B}$ be a set of type $\mathfrak{M}_s^\beta$, and let a_j, $a_j \in B^j$, $j = 1, \dots, s$, be fixed points. There exists a constant $N > 0$ such that, for every m.c. $\omega(x)$ satisfying, for some $\sigma \ge 1$ and $\gamma > 0$, the condition $\omega(\lambda x) \le \sigma\lambda^\gamma\omega(x)$ (for all $\lambda \ge 1$ and $x > 0$), the inclusion $\omega(x) \in \{\overline{B}, \sigma N^\gamma\}$ holds. Moreover, if $\Omega_0(x) < \infty$, then $\Omega_0(x) \in \{\overline{B}, \sigma N^\gamma\}$.

In the proof of this lemma the following will be used.

Lemma 2. Let D be a simply connected domain of the extended plane; ξ_0 a finite point of D; h the distance from ξ_0 to ∂D. If the diameter of the set-

of the set ∂D is not less than $\delta > 0$, then on the circle $|\zeta - \zeta_0| = h$

$$g_D(\zeta,\zeta_0) \le \ln\left[3 + 8h/\delta + 2\sqrt{2(1+4h/\delta)(1+2h/\delta)}\right] \equiv g(h/\delta),$$

where $g_D(\zeta, \zeta_0)$ is the generalized Green function of the domain D.

Proof of Lemma 2. On ∂D there exist points ζ_1 and ζ_2 such that $|\zeta_1 - \zeta_0| = h$ and $|\zeta_1 - \zeta_2| = \frac{1}{2}\delta$. Under the transformation

$$w = w(\zeta) = (\zeta_1 - \zeta_2)(\zeta - \zeta_0)[4(\zeta_1 - \zeta_0)(\zeta - \zeta_2)]^{-1}$$

the domain D is mapped onto a finite domain D_w containing the point $w = 0$ and not containing the point $w = \frac{1}{4}$. Let $t = t(w)$, $t(0) = 0$, be the one-sheeted conformal mapping of the domain D_w onto the unit disk. From the well-known Bieberbach estimates it follows that $|t(w)| \geq 4|w|(1 + \sqrt{1 + 4|w|})^{-2}$. Since on the circle $|\zeta - \zeta_0| = h$ we have $|w(\zeta)| \geq [4(1 + 4h/\delta)]^{-1}$, on the same circle $g_D(\zeta, \zeta_0) = -\ln|t(w(\zeta))| \leq g(h/\delta)$, and Lemma 2 is proved.

Proof of Lemma 1. Let $z_0 \in L$, $u > 0$, and let the point ζ_0 be determined by the condition $|\zeta_0 - z_0| = d(z_0, u)$. If $\zeta_0 \in L$, then $d(\zeta_0, u) \leq 2d(z_0, u)$, and therefore

$$I = \ln[\omega(d(\zeta_0, u))/\omega(d(z_0, u))] \leq \ln[\sigma \cdot 2^\gamma]. \tag{4}$$

Next, let $\zeta_0 \notin L$. Denote by d the diameter of the set L, and by $d(u)$ the distance from L to L_{1+u}. It is clear that $d(u)$, $u > 0$, is an increasing function and $d(u) \leq d(z_0, u) \leq d + d(u)$, and therefore

$$I \leq \ln[\omega(d + d(u))/\omega(d(u))] \leq \ln[\sigma(1 + d/d(u))^\gamma]. \tag{5}$$

Choose $u_0 > 0$ so that $\sup d(z, u_0) \leq \frac{1}{4}\beta$. Further suppose that $0 < u < u_0$. Let $\zeta_0 \in B^j$. We note that for $z \in \partial B^j$, $|z - \zeta_0| \geq d(z_0, u)$, we have

$$\omega(d(z, u)) \leq \omega(3|z - \zeta_0|) \leq \sigma(3|z - \zeta_0|/|z_0 - \zeta_0|)^\gamma \omega(d(z_0, u)),$$

and for $z \in \partial B^j$, $|z - \zeta_0| \leq d(z_0, u)$, we have $\omega(d(z, u)) \leq \omega(3|z_0 - \zeta_0|) \leq \sigma \cdot 3^\gamma \omega(d(z_0, u))$. Introduce the function $\varphi(z) = |z - \zeta_0|$ for $z \in \partial B^j$, $|z - \zeta_0| \geq d(z_0, u)$, and $\varphi(z) = |z_0 - \zeta_0|$ for $z \in \partial B^j$, $|z - \zeta_0| \leq |z_0 - \zeta_0|$. Then

$$\begin{aligned} H_j(\zeta, \ln\omega(d(z, u))) &\leq H_j\left(\zeta, \ln[\sigma(3/|z_0 - \zeta_0|)^\gamma \varphi(z)^\gamma \omega(d(z_0, u))]\right) \\ &= \ln[\sigma \cdot 3^\gamma] + \gamma[H_j(\zeta, \ln\varphi(z)) - \ln|\zeta_0 - z_0|] + \ln\omega(d(z_0, u)), \end{aligned}$$

and hence

$$I \leq \ln[\sigma \cdot 3^\gamma] + \gamma[H_j(\zeta_0, \ln\varphi(z)) - \ln|\zeta_0 - z_0|]. \tag{6}$$

Denote by $\mathcal{K}$ the disk $|\zeta - \zeta_0| < |z_0 - \zeta_0|$. It is easy to see that

$$H_j(\zeta, \ln\varphi(z)) \le g_{B^j \cup \mathcal{K}}(\zeta, \zeta_0) + \ln|\zeta - \zeta_0|, \qquad \zeta \in B^j,$$

where $g_{B^j \cup \mathcal{K}}(\zeta, \zeta_0)$ is the generalized Green function of the domain $B^j \cup \mathcal{K}$. Since the right-hand side of the last inequality is harmonic in $\mathcal{K}$, it follows that

$$H_j(\zeta_0, \ln\varphi(z)) \le \sup_{\zeta \in \partial\mathcal{K}} g_{B^j \cup \mathcal{K}}(\zeta, \zeta_0) + \ln|z_0 - \zeta_0|.$$

Hence, from (6) we have

$$I \le \ln[\sigma \cdot 3^\gamma] + \gamma \sup_{\zeta \in \partial\mathcal{K}} g_{B^j \cup \mathcal{K}}(\zeta, \zeta_0). \tag{7}$$

From the preceding it is clear that there exists a continuum $L^* \in L$, separated from the point ζ_0 by the distance $|z_0 - \zeta_0|$ and having diameter not less than the number $|z_0 - \zeta_0|$. If by D we denote that one of the domains complementary to L^* which contains $B^j \cup \mathcal{K}$, then $g_{B^j \cup \mathcal{K}}(\zeta, \zeta_0) \le g_D(\zeta, \zeta_0)$, and on $\partial\mathcal{K}$, by Lemma 2 $(h/\delta \le 1)$, we have $g_D(\zeta, \zeta_0) \le \ln 22$. Thus (see (7)), $I \le \ln[\sigma \cdot 66^\gamma]$. Hence, from (4) and (5) we conclude that $\omega(x) \in \{\overline{B}, \sigma N^\gamma\}$, where $N = \max\{1 + d/d(u_0), 66\}$, and Lemma 1 is proved.

Remark 5. Denote by N_* the exact lower bound of the numbers N such that, for any m.c. $\omega(x)$ satisfying the conditions of Lemma 1, the inclusion $\omega(x) \in \{\overline{B}, \sigma N^\gamma\}$ holds. Obviously, N^* depends only on $\overline{B}$ and the points a_j, $j = 1, \dots, 5$. If in the formulation of the theorem $\overline{B}$ is a set of type $\mathfrak{M}_\beta^s$, $\omega(x)$ satisfies the conditions of Lemma 1, then for M the estimate $M \le \sigma N_*^{r+\gamma}$ holds. The constants $K_\omega^{(p)}$, $p \ge -1$, are easily estimated from above:

$$K_\omega^{(p)} \le \sigma \left(\int_{e^{-1}}^{1} \tau^{p+\gamma-1}\, d\tau \right)^{-1}.$$

At the same time, by Lemma 1, in the formulation of the theorem the condition $x^r \omega(x) \in \{\overline{B}, M\}$ becomes superfluous.

Received
30 V 1967

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Note: Figure translations are in progress. See original paper for figures.