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MAPPINGS OF TYPE s OF LOCALLY CONVEX SPACES

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In the first part of the present note, the concept of mappings of type s of normed spaces, which were considered by Grothendieck ([4]) (under the name of Fredholm mappings of zero order) and by Pietsch (([5]), 8.5), is carried over to locally convex spaces. At the same time, the basic properties of mappings of type s of normed spaces established in ([1]) are preserved also for mappings of type s of locally convex spaces. In the second part, a number of examples of strongly nuclear function spaces are given.

1. **Definition 1.** Let E be a separated locally convex space and let E' be its dual. A sequence (x_n) in E (respectively (x'_n) in E') will be called **locally rapidly decreasing** if (x_n) (respectively (x'_n)) decreases rapidly* in the normed space E_A (respectively E'_B) for some bounded set A in E (respectively some equicontinuous set B in E'); a set C in E (respectively in E') will be called **strongly nuclear** if C is contained in the closed (respectively in the $\sigma(E', E)$-closed) absolutely convex hull of some locally rapidly decreasing sequence in E (respectively in E').

Definition 2. Let E and F be separated locally convex spaces. A linear mapping $f : E \to F$ will be called a **mapping of type** s if the image of some neighborhood of zero in E is a strongly nuclear set in F.

In my note ([1]), Theorem 3b, it was shown that a mapping f of a normed space X into a normed space Y is a mapping of type s if and only if the image of the unit ball of X is a strongly nuclear set in Y. Thus, for the case of normed spaces, the definition given above and the usual definition of mappings of type s are equivalent.

Let E, F be separated locally convex spaces and let t_{sN} be the strongest strongly nuclear topology** in E, majorized by the original one.

Theorem 1. The following conditions for a mapping $f \in \mathcal{L}(E, F)$ are equivalent:

a) f is a mapping of type s;

b) $f : (E, t_{sN}) \to F$ is bounded (i.e., the image of some neighborhood of zero is a bounded set);

c) there exist an equicontinuous sequence (x'_n) in E', a bounded sequence (y_n) in F, and a rapidly decreasing numerical—

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* A sequence (x_n) in a locally convex space X is called rapidly decreasing if $n^k x_n \to 0$ in X for every k; a set $A \subset X$ is called nuclear if A is contained in the closed absolutely convex hull of some rapidly decreasing sequence from X (see ([1])).

** A locally convex space X is called strongly nuclear ([1]) if for every neighborhood of zero U there exists a neighborhood of zero $V \subset U$ such that the canonical embedding $X'_{U^0} \to X'_{V^0}$ is a mapping of type s.

a numerical sequence (λ_n) such that

$$f(x) = \sum_N \lambda_n \langle x, x'_n \rangle y_n \quad \text{for all } x \in E;$$

c) there exists a locally rapidly decreasing sequence (x'_n) in E' such that

$$p(f(x)) \leq \lambda_p \sum_N |\langle x, x'_n \rangle|$$

for every continuous seminorm p on F and all $x \in E$, where λ_p depends only on p;

d) there exist a strongly nuclear set M in E' and a positive Radon measure μ on M such that

$$p(f(x)) \leq \lambda_p \int_M |\langle x, x' \rangle| \, d\mu$$

for all $x \in E$ and every continuous seminorm p on F, where λ_p depends only on p.

Proposition 1. The mapping adjoint to a mapping of type s is also a mapping of type s.

A locally convex space X is called σ-quasi-barreled ([5]) if every strongly bounded sequence in X' is equicontinuous.

Proposition 2. Let E be σ-quasi-barreled, and let F be arbitrary locally convex spaces and $f \in L(E, F)$. If ${}^t f$ is a mapping of type s, then f is also a mapping of type s.

Proposition 3. In a metrizable space every nuclear set is strongly nuclear.

Corollary. Let E be arbitrary, and let F be metrizable locally convex spaces. A mapping $f \in \mathcal{L}(E, F)$ is a mapping of type s if and only if the image of some neighborhood of zero from E is a nuclear set in F.

We shall give a characterization of strongly nuclear spaces by means of mappings of type.

Proposition 4. A locally convex space is nuclear if and only if every continuous mapping of it into a Banach space is a mapping of type s.

2. We give some examples of strongly nuclear spaces.

Let G be an open subset in $R \times R$, and let $\mathcal{H}(G)$ be the space of all harmonic functions on G with the topology of compact convergence.

Theorem 2. The space $\mathcal{H}(G)$ is strongly nuclear.

Let $(m_{pq})_{(p;q)\in N\times N}$ be a numerical matrix such that $1 \le m_{0q} \le m_{1q} \le \cdots \le m_{pq} \le \cdots$ for each q, and let $k(m_{pq})$ be the set of all numerical sequences (x_q) for which $\sup_q |m_{pq}x_q| < \infty$ for every p. The space $k(m_{pq})$ becomes countably normed if one sets $\|(x_q)\|_p = \sup_q |m_{pq}x_q|$.

Theorem 3. The space $k(m_{pq})$ is strongly nuclear if and only if, for every p_1, there exists p_2 such that the sequence

$$\left(\frac{m_{p_1 q}}{m_{p_2 q}}\right)_q$$

is rapidly decreasing.

Now let $(M_p)_{p\in N}$ be a sequence of continuous functions on the line, increasing at each point: $1 \le M_0(x) \le M_1(x) \le \cdots \le M_p(x) \le \cdots$ for each $x \in R$. By $\mathcal{K}(M_p)$ (see ([3])) is denoted the space of all infinitely differentiable functions on the -

line, for which

$$\max_{0\le r\le p} \sup_{x\in R} \left|M_p(x)\varphi^{(r)}(x)\right| < \infty.$$

This space becomes countably normed if one sets

$$\|\varphi\|_p = \max_{0\le r\le p} \sup_{x\in R} \left|M_p(x)\varphi^{(r)}(x)\right|.$$

Theorem 4. If for each $p \in N$ there is a $q \in N$ such that

$$\frac{M_p(x)}{M_q(x)}\,|x|^k \longrightarrow 0 \quad \text{as } |x| \to \infty$$

for any k, then the space $\mathcal{K}(M_p)$ is strongly nuclear.

In particular, if

$$M_p(x) = \exp\left[a\left(1 - \frac{1}{p}\right)|x|^{1/\alpha}\right],$$

where $a, \alpha > 0$ (the space $\mathcal{K}(M_p)$ with this sequence (M_p) is denoted by S_α (see ([3])), then it is easy to see that the sequence M_p satisfies the condition of Theorem 4. Therefore we have

Corollary. The space S_α is strongly nuclear.

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Note: Figure translations are in progress. See original paper for figures.