CYBERNETICS AND CONTROL THEORY

A. Kh. GELIG

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Abstract

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A. Kh. GELIG

STABILITY OF NONLINEAR IMPULSE SYSTEMS

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As one of the topical problems awaiting solution, E. Jury in his report at the Third All-Union Conference on Automatic Control ([1]) singled out the study of the dynamics of systems with pulse-width and frequency-pulse modulation ([2]). Below, sufficient frequency conditions for stability and dissipativity of systems of this kind are formulated.

1. Consider a nonlinear impulse system whose mathematical description has been reduced to the equations

$$\sigma_i(t) = f_i(t) + f_i^0 - \sum_{j=1}^{l} \int_0^t [\gamma_{ij}(t-\lambda) + \rho_{ij}]\eta_j(\lambda)\, d\lambda; \tag{1}$$

$$\eta_i(t) = \begin{cases} 0, & \text{if } |\sigma_{i,n}| \le \Delta_i, \quad t_{i,n} < t \le t_{i,n+1}, \\ s_{i,n}(t) \operatorname{sign} \sigma_{i,n}, & \text{if } |\sigma_{i,n}| > \Delta_i, \quad t_{i,n} < t \le t_{i,n+1}; \end{cases} \tag{2}$$

$$t_{i,n} = t_{i,n-1} + T_{i,n}, \qquad t_{i,0} = 0, \qquad n = 1, 2, \dots \tag{3}$$

$$(i = 1, \dots, l; \qquad \sigma_{i,n} \equiv \sigma_i(t_{i,n} - 0)).$$

Here the constants f_i^0 and the functions $f_i(t)$ represent the natural oscillations of the linear part of the system; $\sigma_i(t)$ is the signal at the input of the i-th impulse element (IE$_i$), and $\eta_i(t)$ is the signal at its output.

The function $s_{i,n}(t)$ describes the shape of the pulse produced by IE$_i$ at the instant $t_{i,n}$, and is assumed to be piecewise continuous, nonnegative for $t_{i,n} < t \le t_{i,n+1}$, and equal to zero for $t \le t_{i,n}$, $t > t_{i,n+1}$. The nonnegative constant Δ_i characterizes the size of the dead zone of IE$_i$. The quantity $T_{i,n}$ may be either a nonlinear function of $|\sigma_{i,n}|$ or a functional of $\sigma_i(t)$ ([3]); moreover, in the latter case the value of $T_{i,n}$ depends on the behavior of the function $\sigma_i(t)$ only for $t < t_{i,n}$. It is assumed that there exist positive constants ν_*, ν^* such that

$$1/\nu_* < 1/T_{i,n} < \nu^* \qquad (i = 1, \dots, l;\ n = 1, 2, \dots), \tag{4}$$

i.e., the pulse frequency is bounded from below and from above. Formulas (2), (3) encompass all types of modulation with finite pulse duration described in ([2]): amplitude, width, and time (first- and second-order frequency and phase) modulation.

Let us define some characteristics of the impulse elements. Let $\tau_{i,n}$ be the pulse duration (the measure of the set of values $t \in (t_{i,n}; t_{i,n+1}]$ for which $s_{i,n}(t) > 0$),

$$m_{i,n} = \frac{1}{\tau_{i,n}} \int_{t_{i,n}}^{t_{i,n+1}} s_{i,n}(t)\, dt, \qquad a_{i,n} = \frac{1}{\tau_{i,n} m_{i,n}} \int_{t_{i,n}}^{t_{i,n+1}} s_{i,n}^2(t)\, dt,$$

$$\varkappa_{i,n} = \frac{2}{m_{i,n}\tau_{i,n}|\sigma_{i,n}|} \int_{t_{i,n}}^{t_{i,n+1}} (t - t_{i,n}) s_{i,n}(t)\, dt, \qquad m_i = \inf_{|\sigma_{i,n}| > \Delta_i} m_{i,n},$$

$$M_i = \sup_{|\sigma_{i,n}| > \Delta_i} \left\{ \max_{t \in [t_{i,n}; t_{i,n+1}]} |s_{i,n}(t)| \right\}, \qquad \varkappa_i = \sup_{|\sigma_{i,n}| > \Delta_i} \varkappa_{i,n},$$

$$\alpha_i = \sup_{|\sigma_{i,n}| > \Delta_i} \alpha_{i,n}.$$

If IE_i produces rectangular pulses of height a_i, then, evidently, $m_{i,n} = \alpha_{i,n} = M_i = m_i = \alpha_i = a_i$, while the characteristic $\varkappa_i$ coincides with the "steepness" characteristic of the impulse element introduced in (2).

It is assumed that the estimates

$$M_i < \infty, \qquad m_i > 0 \qquad (i = 1, \dots, l), \tag{5}$$

hold, that the functions $f_i(t)$ are absolutely continuous for $t > 0$, and that the relations

$$f_i(t) \in L_1[0, \infty), \qquad \lim_{t \to \infty} \dot{f}_i(t) = 0 \qquad (i = 1, \dots, l); \tag{6}$$

$$\rho_{ij} = \text{const}, \qquad \gamma_{ij}(t) \in L_1[0, \infty) \cap L_2[0, \infty) \qquad (i, j = 1, \dots, l); \tag{7}$$

$$\rho_{ij} + \gamma_{ij}(t) = g_{ij}(t) + \sum_{k=1}^{\infty} c_{i,j}^{(k)}\, 1(t - \lambda_{i,j}^{(k)}), \tag{8}$$

are satisfied, where the functions $g_{ij}(t)$ are absolutely continuous for $t > 0$, and

$$\dot{g}_{ij}(t) \in L_1[0, \infty), \qquad \sum_{k=1}^{\infty} |c_{i,j}^{(k)}| < \infty, \tag{9}$$

$$\lambda_{i,j}^{(k)} = \text{const} \geq 0, \qquad 1(t) = 0 \text{ for } t < 0, \qquad 1(t) = 1 \text{ for } t \geq 0.$$

The last requirement is due to the circumstance that the functions $\gamma_{ij}(t)$ may be discontinuous if the linear part of the control system contains elements with distributed parameters.

We shall assume that the functions

$$\chi_{ij}(p) = \int_0^\infty \gamma_{ij}(t) \exp(-pt)\, dt$$

are analytic for $\operatorname{Re} p > 0$, and introduce the notation

$$r_i = \sum_{j=1}^{l} \left(|g_{ij}(+0)| + \int_0^\infty |\dot{g}_{i,j}(t)|\, dt + \sum_{k=1}^{\infty} |c_{i,j}^{(k)}| \right) M_j, \qquad k_i = \frac{r_i \Delta_i \delta_i}{2\alpha_i};$$

K and D are diagonal matrices with elements $k_1, \dots, k_l$ and $d_1, \dots, d_l$, respectively*; $\Gamma(p)$ is the square matrix with elements $\chi_{ij}(p)$; R is the square matrix with elements ρ_{ij}.

Theorem 1. *Let the following conditions be satisfied:*

1) *relations (4)–(9) hold;*
2) *there exist positive constants δ_i such that*

$$\varkappa_i < 2/r_i - \delta_i \qquad (i = 1, \dots, l);$$

3) *if $\Delta_i > 0$, then*

$$\inf_{|\sigma_{i,n}| > \Delta_i} \tau_{i,n} > 0;$$

   if $\Delta_i = 0$, then the dependence of $\tau_{i,n}$ on $\sigma_{i,n}$ is such that

$$\lim_{\tau_{i,n} \to 0} \sigma_{i,n} = 0;$$

4) *there exist positive constants $d_1, \dots, d_l$ such that the matrix DR is symmetric and nonnegative and, for all real ω, the matrix*

$$Q(\omega) = DK + 0.5(D\Gamma(i\omega) + \Gamma^*(i\omega)D) \qquad (i = \sqrt{-1});$$

   is nonnegative;

5) *either* $\det R \neq 0$*, or the constants f_i^0 are such that the system of equations*

$$f_i^0 = \sum_{j=1}^{l} \rho_{ij} u_j \qquad (i = 1, \dots, l)$$

   is solvable with respect to $u_1, \dots, u_l$.

* The numbers δ_i and d_i will be described below.

Then the solution of system (1) has the following properties:

1)

$$\lim_{r_0 \to 0} \left[\max_i \left(\sup_{t>0} |\sigma_i(t)| \right) \right] = 0,$$

where

$$r_0 = \sum_{i=1}^{l} \left(\sup_{t>0} |f_i(t)| + |f_i^0| + \sup_{t>0} |\dot{f}_i(t)| + \int_0^\infty |f_i(t)|\, dt \right); \tag{10}$$

2) $\lim_{n\to\infty} \sigma_{i,n} = 0$, if $\Delta_i = 0$;

3) if $\Delta_i > 0$, then there exists a $T_* > 0$ such that $|\sigma_i(t_{i,n} - 0)| \le \Delta_i$ for all $t_{i,n} > T_*$.

Condition 2) of Theorem 1 cannot be substantially weakened, since there exists an example for which $\chi_i = 2/r_i$ and all the remaining conditions of Theorem 1 are satisfied, but its conclusions do not hold.

Theorem 1 covers the critical cases when the characteristic equation of the linear part of the system has no more than l zero roots. Let us now consider the noncritical case. In this case, in equations (1) $f_i^0 = \rho_{ij} = 0$ $(i, j = 1, \dots, l)$, and the following assertion holds.

Theorem 2. *Suppose that in equations (1) and expressions (8), (10) $f_i^0 = \rho_{ij} = 0$ $(i, j = 1, \dots, l)$, conditions 1) and 2) of Theorem 1 are satisfied, and there exist positive constants $d_1, \dots, d_l$ such that the matrix $Q(\omega)$ is nonnegative for all real ω.*

Then $\lim_{t\to\infty} \sigma_i(t) := 0$ $(i = 1, \dots, l)$ *and*

$$\lim_{r_0 \to 0} \left[\max_i \left(\sup_{t>0} |\sigma_i(t)| \right) \right] = 0.$$

2. Let us now consider the system of equations

$$\sigma_i(t) = f_i(t) + f_i^0 - \int_0^t [\gamma_{ij}(t-\lambda) + \rho_{ij}] \eta_j(\lambda)\, d\lambda + \psi_i(t); \tag{11}$$

$$\eta_i(t) = \sum_{n=1}^{\infty} \lambda_{i,n} \delta(t - t_{i,n}); \tag{12}$$

$$\lambda_{i,n} = \begin{cases} 0, & \text{if } \varphi_i(t_{i,n}-0)=0, \\ \operatorname{sign}\varphi_i(t_{i,n}-0), & \text{if } \varphi_i(t_{i,n}-0)\neq 0 \end{cases} \tag{13}$$

$$(i=1,\dots,l).$$

Here $\sigma_i(t)$, $f_i(t)$, $f_i^0 = \text{const}$, $\gamma_{ij}(t)$, $\rho_{ij} = \text{const}$, $t_{i,n}$ have the same meaning as in equations (1); $\psi_i(t)$ are constantly acting disturbances; $\delta(t)$ is the Dirac delta function (4), and the instants $t_{i,n}$ are determined by formula (3), with $T_{i,n} > 1/\nu^*$.

Each function $\varphi_i(t)$ is piecewise continuous and is the value, at $\sigma_i(t)$, of some nonlinear operator A_i. The operators A_i are defined on piecewise-continuous functions and satisfy the following conditions:

1) $\varphi_i(t-0)$ depends on the values of the function $\sigma_i(\tau)$ only for $0 \le \tau \le t-0$;
2) $\varphi_i(t-0)\sigma_i(t-0) > 0$ for those instants t at which $\sigma_i(t-0) \notin [-\Delta_i'', \Delta_i']$, where Δ_i'', Δ_i' are nonnegative constants.

Equations (11)–(13) describe a nonlinear impulsive system with l impulsive elements performing frequency modulation by instantaneous impulses (5).

It is assumed that the functions $\gamma_{ij}(t)$ are continuous for $t > 0$, belong to $L_2[0,\infty)$, and their Laplace transforms $\chi_{ij}(p)$ are analytic for $\operatorname{Re} p > 0$. With respect to the functions $f_i(t)$ and $\psi_i(t)$ we shall assume that they are continuous for $t > 0$ and that, whatever the sequence $t_1, t_2, \dots$ satisfying the condition $t_{n+1} - t_n > 1/\nu^*$, the corres-

relations

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |f_i(t_k)| = 0, \qquad \overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\psi_i(t_k)| < \infty \quad (i=1,\dots,l).$$

From equations (11) it is seen that at the instant $t_{j,k}$ the functions $\sigma_i(t)$ undergo a jump of magnitude $-(\gamma_{ij}(+0)+\rho_{ij})\lambda_{j,k}$, which, generally speaking, does not tend to zero as $k\to\infty$, even if $\Delta_i'' = \Delta_i' = \psi_i(t) \equiv 0$ $(i=1,\dots,l)$. Therefore, instead of the problem of stability in the large, we shall study the dissipativity of system (11)–(13).

Theorem 3. *Suppose that system (11)–(13) satisfies the assumptions made in this section and that conditions 4) (for $K=0$) and 5) of Theorem 1 are fulfilled. Then, for a solution of system (11)–(13), the estimate*

$$\overline{\lim}_{T\to\infty} \|\sigma\|_T \leqslant 2\Delta + \gamma + \overline{\lim}_{T\to\infty} \|\psi\|_T, \tag{14}$$

is valid, where

$$\Delta = \sum_{i=1}^{l} d_i \max(\Delta_i', \Delta_i''), \qquad \gamma = l \sum_{i=1}^{l} d_i \sum_{j=1}^{l} (|\gamma_{ij}(+0)| + |\rho_{ij}|),$$

$$\|\sigma\|_T = \frac{1}{N} \sum_{i=1}^{l} d_i \sum_{0<t_{i,k} \leqslant T} |\sigma_i(t_{i,k} - 0)|, \qquad N = \max_{\substack{t_{i,k} \leqslant T \\ i=1,\dots,l}} k,$$

$$\|\psi\|_T = \frac{1}{N} \sum_{i=1}^{l} d_i \sum_{0<t_{i,k} \leqslant T} |\psi_i(t_{i,k})|.$$

There exists an example satisfying the conditions of Theorem 3 for which equality is realized in (14).

Suppose that the nonlinear operators A_i satisfy the additional condition*: $\varphi_i(t) \equiv 0$ for those t for which $\sigma_i(t) \in [-\Delta_i'', \Delta_i']$. Then in the right-hand side of inequality (14), Δ may be put in place of 2Δ.

The proof of the theorems formulated above is based on the method of a priori integral estimates, which is applied to estimating a functional quadratic with respect to $\sigma_1, \dots, \sigma_l, \eta_1, \dots, \eta_l$. In the proof of Theorem 3, moreover, the delta-function is first replaced by a delta-like sequence, and at the end of the exposition a passage to the limit is carried out.

It is interesting to note that the results of this work are applicable in the investigation of the dynamics of mathematical models of neural networks.

Leningrad State University
named after A. A. Zhdanov

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* This case occurs in the presence of insensitivity in pulse elements.

Note: Figure translations are in progress. See original paper for figures.