SOME LIMIT THEOREMS FOR LARGE DEVIATIONS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 519.21

MATHEMATICS

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SOME LIMIT THEOREMS FOR LARGE DEVIATIONS

(Presented by Academician Yu. V. Linnik on 12 VII 1967)

1. Consider a sequence of independent random variables $X_1, X_2, \dots$ with finite variances $\sigma_1^2, \sigma_2^2, \dots$, not all of which are equal to zero, and such that $EX_j = 0$ $(j = 1, 2, \dots)$. Put

$$B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad Z_n = \frac{1}{B_n} \sum_{j=1}^{n} X_j,$$

$$F_n(x) = P\{Z_n < x\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

In the case of identical distributions, Cramér ([1]) obtained a theorem on the probabilities of large deviations of the quantity Z_n, which was generalized by V. V. Petrov ([2]) to the case of non-identically distributed variables, with a simultaneous strengthening of it also for the particular case of identical distributions. We give the statement of Cramér' s theorem, taking this strengthening into account.

If $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables and if, for some $a > 0$,

$$Ee^{a|X_1|} < \infty, \tag{1}$$

then, for $x > 1$, $x = o(\sqrt{n})$, and $n \to \infty$, we have

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}} \lambda\left(\frac{x}{\sqrt{n}}\right)\right\} \left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left\{-\frac{x^3}{\sqrt{n}} \lambda\left(-\frac{x}{\sqrt{n}}\right)\right\} \left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where

$$\lambda(t) = c_0 + c_1 t + c_2 t^2 + \cdots \tag{2}$$

is a power series (Cramér' s series), convergent for all sufficiently small $|t|$.

Analogous results under assumption (1) were obtained by W. Richter ([3]) for local theorems.

Yu. V. Linnik ([4–7]) and V. V. Petrov ([8], [9]) obtained a number of integral and local limit theorems for the case where Cramér' s condition (1) is not fulfilled.

2. If s is a nonnegative integer, then $\lambda_n^{[s]}(t)$ will denote the segment of the series $\lambda_n(t)$ consisting of its first $s+1$ terms:

$$\lambda_n^{[s]}(t) = \sum_{k=0}^{s} c_{kn} t^k.$$

The series $\lambda_n(t)$ is defined in the work of V. V. Petrov ([10]). In the particular case of identical distributions this series coincides with Cramér' s series $\lambda(t)$.

Theorem 1. Suppose that, for some α, $0 < \alpha < 1/2$, the conditions

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E \exp\{|X_j|^{4\alpha/(2\alpha+1)}\} < \infty, \tag{3}$$

$$\overline{\lim}_{n\to\infty} \frac{B_n^2}{n} > 0. \tag{4}$$

are satisfied. Then

$$\frac{1-F_n(x)}{1-\Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}} \lambda_n^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right\} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{5}$$

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left\{-\frac{x^3}{\sqrt{n}} \lambda_n^{[s]}\left(-\frac{x}{\sqrt{n}}\right)\right\} \left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right] \tag{6}$$

as $n \to \infty$ in the region $0 \le x \le n^\alpha/\rho(n)$, where $\rho(n)$ is an arbitrary function satisfying the condition

$$\lim_{n\to\infty} \rho(n) = +\infty. \tag{7}$$

Here s is a nonnegative integer determined by the inequalities

$$s/2(s+2) < \alpha \leq s + 1/2(s+3). \tag{8}$$

We give one of the simplest corollaries of this theorem. Suppose that the conditions of Theorem 1 are satisfied for $0 < \alpha \leq 1/6$. Then

$$\lim_{n\to\infty} \frac{1 - F_n(x)}{1 - \Phi(x)} = 1, \qquad \lim_{n\to\infty} \frac{F_n(-x)}{\Phi(-x)} = 1$$

in the region $0 \leq x \leq n^\alpha/\rho(n)$, whatever the function $\rho(n)$ satisfying condition (7) may be.

Theorem 2. Suppose

$$\overline{\lim}_{n\to\infty} \frac{B_n^2}{n} < \infty.$$

Suppose, further, that there exist positive constants $\alpha < 1/2$, b_0 and b_1, and a function $\rho(n)$ satisfying condition (7), such that

$$1 - F_n(x) \leq b_0 e^{-b_1 x^2}, \qquad F_n(-x) \leq b_0 e^{-b_1 x^2} \tag{9}$$

for $0 \leq x \leq n^\alpha \rho(n)$ and all sufficiently large n. Then

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E \exp\{|X_j|^{4\alpha/(2\alpha+1)}\} < \infty.$$

Let us note that inequalities (9) are satisfied in the region $0 \leq x \leq n^\alpha \rho(n)$ for a sufficiently slowly increasing function $\rho(n)$, if relations (5) and (6) hold in this region and if the function $\left|\lambda_n^{[s]}(x/\sqrt{n})\right|$ is bounded.

3. Introduce the notation

$$v_j(t) = Ee^{itX_j}, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Denote by $p_n(x)$ the derivative of the distribution function $F_n(x)$, if $F_n(x)$ is absolutely continuous.

Theorem 3. Suppose that, for some α, $0 < \alpha < 1/2$, the conditions

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E \exp\{|X_j|^{4\alpha/(2\alpha+1)}\} < \infty, \tag{10}$$

$$\overline{\lim}_{n\to\infty}\frac{B_n^2}{n} > 0.$$

are satisfied. Suppose, further, that to every $\varepsilon > 0$ there corresponds such a $\delta > 0$ that

$$\int_{|t|>\varepsilon}\prod_{j=1}^{n}|v_j(t)|\,dt = O\left(e^{-\delta n^{2\alpha}}\right) \qquad (n\to\infty).$$

Then, for all sufficiently large n, there exists an everywhere continuous density $p_n(x)$ of the distribution of the random variable Z_n, and, moreover,

$$\frac{p_n(x)}{\varphi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}}\,\lambda_n^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1+O\left(\frac{|x|+1}{\sqrt{n}}\right)\right] \tag{11}$$

as $n\to\infty$ in the region $|x|\le n^\alpha/\rho(n)$, where $\rho(n)$ is an arbitrary function satisfying condition (7), and s is the nonnegative integer determined by inequalities (8).

Corollary. Suppose the conditions of Theorem 3 are fulfilled for $0<\alpha\le 1/6$. Then

$$\lim_{n\to\infty}\frac{p_n(x)}{\varphi(x)} = 1$$

in the region $|x|\le n^\alpha/\rho(n)$, whatever the function $\rho(n)$ satisfying condition (7).

Theorem 4. Suppose

$$\overline{\lim_{n\to\infty}}\,\frac{B_n^2}{n} < \infty$$

and the random variable Z_n, for some $n = n_0$, has an absolutely continuous distribution with density $p_n(x)$. Suppose, furthermore, that there exist positive constants $\alpha < 1/2$, b_0, and b_1, and a function $\rho(n)$ satisfying condition (7), such that

$$p_n(x) \le b_0 e^{-b_1 x^2} \tag{12}$$

for $|x|\le n^\alpha\rho(n)$ and all sufficiently large n. Then condition (3) is fulfilled.

4. In papers ([8], [9]) results close to Theorems 1 and 3 were obtained under the assumption that a condition stronger than (3) is fulfilled, namely,

$$E \exp\left\{|X_j|^{4\alpha/(2\alpha+1)}\right\} \leq C \qquad (j = 1, 2, \ldots), \tag{13}$$

where C is some constant. In addition, we have obtained more precise estimates of the remainder terms.

Theorems 2 and 4 are strengthenings of the corresponding theorems from ([8], [9]).
I express my deep gratitude to V. V. Petrov for the formulation of the problem and for valuable suggestions.

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Received
29 VI 1967

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