# ON ONE DEFINITION OF DIMENSION

MATHEMATICS

1968

SovietRxiv

## Abstract

## Full Text

UDC 513.83

*MATHEMATICS*

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# ON ONE DEFINITION OF DIMENSION

*(Presented by Academician P. S. Aleksandrov on 8 IV 1967)*

In dimension theory, as is known, there are two basic definitions of the dimension of topological spaces:

1. The definition of the dimension of a space $X$ by means of coverings; it leads to the invariant $\dim X$.
2. The inductive definition, leading to two invariants: $\operatorname{ind} X$ and $\operatorname{Ind} X$.

For metrizable spaces with a countable base all three invariants coincide, but for broader classes of spaces the question of the relations among them becomes more complicated. For normal spaces, in particular for bicompacta, certain inequalities between these invariants and examples of noncoincidence are known. In the present paper a new variant of the definition of dimension, proposed by A. V. Arkhangel' skii, is investigated. This definition combines features of the definition of dimension by means of coverings and of the inductive definition.

**Definition.** For the empty set $\Lambda$ we put $\operatorname{Dind} \Lambda = -1$. Assuming that the inequality $\operatorname{Dind} X \leq n-1$ has a known meaning, we say that $\operatorname{Dind} X \leq n$ if into every finite covering $\gamma = \{G_1, G_2, \dots, G_k\}$ one can insert a system $\omega = \{B_1, B_2, \dots, B_m\}$ of open pairwise disjoint sets such that

$$\operatorname{Dind}\left(X \setminus \bigcup_{i=1}^{m} B_i\right) \leq n-1.$$

**Proposition.** If $X$ is a normal space, then

$$\operatorname{Ind} X \leq \operatorname{Dind} X. \tag{1}$$

**Proof.** If the space $X$ is empty, the inequality is obvious. Suppose that the inequality is true for all dimensions $k \leq n-1$, and prove it for dimension $k = n$. Let $\operatorname{Dind} X = n$. Take an arbitrary closed set $F$ and any neighborhood $U$ of it. By the normality of the space $X$ there exists a neighborhood $V$ of the set $F$ whose closure lies entirely in $U$, i.e. $\overline{V} \subseteq U$. Consider the binary covering $\gamma = \{U, X \setminus \overline{V}\}$. By the condition, into it one can insert a system $\omega = \{B_k;\ k = 1, \dots, m\}$ of open pairwise disjoint sets $B_k$ such that the relation

$$\operatorname{Dind}\left(X \setminus \bigcup_{i=1}^{m} B_i\right) \leq n-1$$

holds.

Consider the open set

$$O = V \cup \left(\bigcup_{j=1}^{k} B_{ij}\right),$$

where $B_{ij} \in \omega$ and $B_{ij} \cap \overline{V} \neq \Lambda$; obviously, $F \subseteq O \subseteq U$, and moreover

$$\operatorname{Fr} O \subseteq X \setminus \bigcup_{i=1}^{m} B_i$$

($\operatorname{Fr} O$ is the boundary of the set $O$). By virtue of the heredity of the dimension Dind with respect to closed sets, $\operatorname{Dind}(\operatorname{Fr} O) \leq n-1$, and by the induction hypothesis $\operatorname{Ind}(\operatorname{Fr} O) \leq n-1$. Thus, for an arbitrary closed set $F$ and its arbitrary neighborhood $U$, we have found

neighborhood $O \subseteq U$, the inductive dimension of whose boundary does not exceed $n-1$. This means that $\operatorname{Ind} X \leq n$, whereby inequality (1) is proved.

**Theorem 1.** If $X$ is a perfectly normal space, then

$$\operatorname{Ind} X = \operatorname{Dind} X. \tag{2}$$

**Proof.** In view of the proposition proved, it suffices to prove, for perfectly normal spaces, the inequality

$$\operatorname{Dind} X \leq \operatorname{Ind} X. \tag{3}$$

For the empty space inequality (3) is trivial. Suppose that the inequality is true for $\operatorname{Ind} X = k$, where $k \leq n-1$, and prove (3) for $\operatorname{Ind} X = n$. Consider an arbitrary open finite covering of the space $\gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$.

Construct the closed set

$$F_1 = X \setminus \bigcup_{i=2}^{k} \Gamma_i.$$

Clearly, $F_1 \subseteq \Gamma_1$. Since $\operatorname{Ind} X = n$, there exists a neighborhood $O_1$ of the set $F_1$ such that $O_1 \subseteq \Gamma_1$ and $\operatorname{Ind}(\operatorname{Fr} O_1) \leq n-1$. Note that the system

$\gamma_1 = \{O_1, \Gamma_2, \dots, \Gamma_k\}$ is a covering of the space $X$. Suppose the construction has been carried out for all $i < p$; then for $i = p$ set

$$F_p = X \setminus \left\{ \left( \bigcup_{i=p+1}^{k} \Gamma_i \right) \cup \left( \bigcup_{i=1}^{p-1} O_i \right) \right\}.$$

The set $F_p$ is closed, and $F_p \subseteq \Gamma_p$. Just as for $i = 1$, there is a neighborhood $O_p$ of the set $F_p$ such that $\overline{O_p} \subseteq \Gamma_p$ and $\operatorname{Ind}(\operatorname{Fr} O_p) \leq n - 1$. As a result of the construction we obtain the system $\gamma^* = \{O_1, O_2, \dots, O_k\}$, which is a covering of the space $X$.

By the theorem on the dimension of a finite sum of closed sets, proved by E. Čech ($^1$), we have

$$\operatorname{Ind} \left( \bigcup_{i=1}^{k} \operatorname{Fr} O_i \right) \leq n - 1. \tag{4}$$

Starting from the system $\gamma^*$, construct a system $\eta$ of open pairwise disjoint sets $G_n$. Put $G_1 = O_1$, $G_2 = O_2 \setminus \overline{G_1}$, and, in general,

$$G_n = O_n \setminus \bigcup_{i=1}^{n-1} \overline{G_i}.$$

The system $\eta = \{G_1, G_2, \dots, G_k\}$ consists of pairwise disjoint sets.

The system $\bar{\eta} = \{\overline{G_1}, \overline{G_2}, \dots, \overline{G_k}\}$ is, as is easy to see, a closed covering of the space $X$. Hence we obtain

$$X \setminus \bigcup_{i=1}^{k} G_i \subseteq \bigcup_{i=1}^{k} (\overline{G_i} \setminus G_i) = \bigcup_{i=1}^{k} \operatorname{Fr} G_i,$$

but $G_n = O_n \setminus \bigcup_{i=1}^{n-1} \overline{G_i}$, whence

$$\operatorname{Fr} G_n \subseteq \operatorname{Fr} O_n \cup \operatorname{Fr} \left( \bigcup_{t=1}^{n-1} \overline{G_t} \right) \subseteq \bigcup_{t=1}^{n} \operatorname{Fr} O_t,$$

and, consequently,

$$X \setminus \bigcup_{i=1}^{k} G_i \subseteq \bigcup_{t=1}^{k} \operatorname{Fr} O_t. \tag{5}$$

From (4) and (5), by the hereditary property of the dimension Ind with respect to closed sets, we obtain

$$\operatorname{Ind}\left(X \setminus \bigcup_{i=1}^{k} G_i\right) \leq n-1. \tag{6}$$

From (6), by the induction hypothesis, it follows that

$$\operatorname{Dind}\left(X \setminus \bigcup_{i=1}^{n-1} G_i\right) \leq n-1.$$

Taking into account that the system $\eta$ is inscribed in the original cover, we see that $\operatorname{Dind} X \leq n$, whereby (3), and hence also (2), is proved.

We shall now prove Urysohn' s formula for the dimension Dind in the class of hereditarily normal spaces. For this we shall need the following

**Lemma.** In order that a subset $M$ of a hereditarily normal space $X$ have dimension $\operatorname{Dind} M \leq n$, it is necessary and sufficient that the following condition be fulfilled: for every cover $\gamma = \{\Gamma_i;\ i = 1, 2, \dots, k\}$ of the space $M$ by sets open in $X$, there exists a system $\omega = \{B_i;\ i = 1, 2, \dots, m\}$ of sets open in $X$, with $B_i \cap B_j = \Lambda$ for $i \neq j$, inscribed in $\gamma$ and such that

$$\operatorname{Dind}\left\{M \cap \left(X \setminus \bigcup_{i=1}^{m} B_i\right)\right\} \leq n-1. \tag{7}$$

**Necessity.** Let $\gamma$ be a cover mentioned in the condition. Consider the induced cover $\gamma^*$ of the set $M$, open in $M$:

$$\gamma^* = \{\Gamma_1^*, \Gamma_2^*, \dots, \Gamma_n^*\},$$

where $n \leq k$ and $\Gamma_i^* = \Gamma_i \cap M$. There exists a system

$$\omega^* = \{B_1^*, B_2^*, \dots, B_l^*\},$$

inscribed in $\gamma^*$, with the $B_i^*$ open in $M$ and $B_i^* \cap B_j^* = \Lambda$ for $i \neq j$, and such that

$$\operatorname{Dind}\left(M \setminus \bigcup_{i=1}^{l} B_i^*\right) \leq n-1.$$

We now need one theorem of E. Čech [2], which states that any finite system $\omega^*$ of sets open in some subset $M$ of a hereditarily normal space $X$ admits an extension to a system of sets $\omega' = \{B_i'\}$ open in the space $X$, with $\omega^*$ similar to $\omega'$ and $B_i^* = B_i' \cap M$. It is clear that, by this theorem, $\omega'$ consists of pairwise disjoint sets. Construct the system $\omega = \{B_i\}$, where $B_i = B_i' \cap \Gamma_j$, if $B_i^* \subseteq \Gamma_j^*$. The system $\omega$ is inscribed in the system $\gamma$. Since

$$M \cap \left( X \setminus \bigcup_{i=1}^{l} B_i \right) = M \setminus \bigcup_{i=1}^{l} B_i^*$$

it follows that

$$\operatorname{Dind} \left\{ M \cap \left( X \setminus \bigcup_{i=1}^{l} B_i \right) \right\} \leq n - 1,$$

which was required to be proved.

**Sufficiency.** Suppose the condition of the lemma is fulfilled. Take an arbitrary finite cover $\gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$ of the set $M$ by sets open in $X$. There exists a system $\omega$ of pairwise disjoint sets $B_k$, where $k = 1, 2, \dots, m$, open in $X$, inscribed in $\gamma$ and such that

$$\operatorname{Dind} \left\{ M \cap \left( X \setminus \bigcup_{i=1}^{m} B_i \right) \right\} \leq n - 1.$$

Consider the induced cover $\gamma^* = \{\Gamma_k^*\}$, where $\Gamma_k^* = M \cap \Gamma_k$, and the induced system $\omega^* = \{B_k^*\}$, where $B_k^* = M \cap B_k$. The sets $\Gamma_i^*$, $B_i^*$ are open in $M$, the system $\omega^*$ is inscribed in $\gamma^*$, and $B_i^* \cap B_j^* = \Lambda$ if $i \neq j$,

$$M \setminus \bigcup_i B_i^* = M \cap \left( X \setminus \bigcup_i B_i \right),$$

therefore

$$\operatorname{Dind} \left( M \setminus \bigcup_i B_i^* \right) \leq n - 1,$$

i.e. $\operatorname{Dind} M \leq n$.

It should be noted that the condition of the lemma is sufficient for arbitrary $T_1$-spaces.

With the aid of the lemma one proves

**Theorem 2.** For any two sets $A$ and $B$ of a hereditarily normal space the formula holds

$$\operatorname{Dind}(A \cup B) \leq \operatorname{Dind} A + \operatorname{Dind} B + 1. \tag{8}$$

**Proof.** We shall carry out the proof by double induction on the dimensions of the spaces $A$ and $B$. If $A = B = \Lambda$, then (8) is obvious. Suppose that (8) is true when one of the inequalities (9), (10) is satisfied,

$$\operatorname{Dind} A \leqslant m, \qquad \operatorname{Dind} B \leqslant n-1; \tag{9}$$

$$\operatorname{Dind} A \leqslant m-1, \qquad \operatorname{Dind} B \leqslant n. \tag{10}$$

We prove inequality (8) for $\operatorname{Dind} A = m$ and $\operatorname{Dind} B = n$. Take a cover $\gamma$ of the space $A \cup B$ by sets open in $X$, $\gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$.

Applying the lemma to the set $A$, we conclude that there exists a system $\omega$ of pairwise disjoint sets open in $X$, $\omega = \{B_1, B_2, \dots, B_m\}$, inscribed in $\gamma$ and such that

$$\operatorname{Dind}\left\{A \cap \left(X \setminus \bigcup_{i=1}^{m} B_i\right)\right\} \leqslant m-1. \tag{11}$$

Since

$$B \cap \left(X \setminus \bigcup_{i=1}^{m} B_i\right) \subseteq B,$$

by the heredity of the dimension Dind with respect to closed sets, we have

$$\operatorname{Dind}\left\{B \cap \left(X \setminus \bigcup_{i=1}^{m} B_i\right)\right\} \leqslant n. \tag{12}$$

Taking (11) and (12) into account, one may apply the induction hypothesis to the spaces

$$A \cap \left(X \setminus \bigcup_{i} B_i\right) \quad \text{and} \quad B \cap \left(X \setminus \bigcup_{i} B_i\right).$$

We obtain

$$\operatorname{Dind}\left\{(A \cup B) \cap \left(X \setminus \bigcup_{i} B_i\right)\right\} \leqslant m+n.$$

By the lemma, $\operatorname{Dind}(A \cup B) \leqslant m + n + 1$, as was required to prove. By an obvious induction we obtain the following formula (13) for any finite family of sets $A_1, A_2 \ldots A_{n+1}$ lying in a hereditarily normal space $X$:

$$\operatorname{Dind}\left(\bigcup_{i=1}^{n+1} A_i\right) \leqslant \sum_{i=1}^{n+1} \operatorname{Dind} A_i + n. \tag{13}$$

**Remark.** The heredity of the dimension Dind with respect to closed sets, which we used in the proof of Theorems 1 and 2, is easily proved with the help of the sufficient condition of the lemma in arbitrary $T_1$-spaces.

In conclusion, the authors express their deep gratitude to A. V. Arhangel' skii for posing the problems and for valuable advice during the preparation of the work.

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Received
8 X 1966

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*Note: Figure translations are in progress. See original paper for figures.*