The Occurrence of Discontinuities in the Continuation of Solutions of Nonlinear Mixed Problems for Hyperbolic Equations in the Plane

Mathematics

1968

SovietRxiv

Abstract

Full Text

UDC 517.946

Mathematics

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The Occurrence of Discontinuities in the Continuation of Solutions of Nonlinear Mixed Problems for Hyperbolic Equations in the Plane

(Presented by Academician I. G. Petrovsky, March 6, 1968)

1°. In the strip $0 \leq x \leq 1,\ 0 \leq t < +\infty$ consider the mixed problem

$$u_{xx} - u_{tt} = A(x,t,u)u_x + B(x,t,u)u_t + F(x,t,u), \tag{1}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \quad \text{for } 0 \leq x \leq 1, \tag{2}$$

$$a_0(u)u_x + b_0(u)u_t = f_0(t,u) \quad \text{for } x = 0, \tag{3}$$

$$a_1(u)u_x + b_1(u)u_t = f_1(t,u) \quad \text{for } x = 1. \tag{4}$$

It is known (^{1,2}) that if the function $\dot{u} \in C_2(\overline{\Pi}_{T_0})$ is a solution of problem (1)–(4) in the rectangle

$$\overline{\Pi}_{T_0} = \{0 \leq x \leq 1,\ 0 \leq t \leq T_0\}, \qquad 0 < T_0 < +\infty,$$

and

$$h_i(\dot{u}(i,T_0)) \equiv b_i(\dot{u}(i,T_0)) + (-1)^{i+1} a_i(\dot{u}(i,T_0)) \neq 0 \quad (i = 0,1), \tag{5}$$

then the solution u is continued uniquely into $\overline{\Pi}_{T_0+\Delta T}$ $(\Delta T > 0)$ with preservation of smoothness. Below, problem (1)–(4) is studied in the case when, under continuation of a solution of class C_2, inequality (5) for $i = 0$ turns into an equality at some time $t = T^*,\ 0 < T^* < 1/2.$* The violation of inequality (5) is connected with the occurrence of discontinuous oscillations in a telegraph line, first studied by A. A. Vitt (^{4}).

In the present paper a definition is introduced and theorems of existence and uniqueness of a discontinuous solution (d.s.) of problem (1)–(4) for $t > T^*$

are formulated. Theorem 5 (see 4°) shows that the d.s. of problem (1)–(4) arises in a natural way if one studies the behavior, as $\mu \to 0$, of the solutions of mixed problems obtained by perturbing the operator $a_0(\mathring{u})u_x + b_0(u)u_t$ by the term μu_{tt} $(\mu > 0)$. The necessity of introducing a d.s. for $t > T^*$ is caused by the following circumstance. Consider problem (1)–(4) in $\Pi_{1/2}$. Passing, by the substitution $v = u_x + u_t$, $w = u_x - u_t$, to the mixed problem for the corresponding hyperbolic system and integrating it along characteristics, one can obtain an integral equation for the function $u(x,t)$, equivalent to problem (1)–(4). Such an equation leads to the natural definition of the concept of a continuous generalized solution (c.g.s.) of problem (1)–(4) in $\overline{\Pi}_T$, $0 < T \leq 1/2$. In this case u turns out to be the unique c.g.s. of problem (1)–(4) in $\overline{\Pi}_{T^*}$. However, in the "general" case studied here,** for any $\Delta T > 0$ there is no c.g.s. of problem (1)–(4) in $\overline{\Pi}_{T^*+\Delta T}$, and any increase in the smoothness and compatibility of the data of the problem (the functions $A, B, F, a_i, b_i, f_i, \varphi, \psi$) does not lead to the existence of a continuous solution for $t > T^*$.

Under the assumption $A = B = 0$, a d.s. was considered by us in (^{3}).

2°. Statement of the problem. Smoothness of the function $\mathring{u}$ in $\overline{\Pi}_{T^*}$. 1. Put $R_1 = (-\infty, +\infty)$, $D_0 = \Pi_{1/2} \times R_1$, $D_1 = [0, 1/2] \times R_1$, $h_i = b_i + (-1)^{i+1} a_i$. Let:

1) A, $B \in C_2(D_0)$, $F \in C_1(D_0)$, $f_i \in C_1(D_1)$, a_i, $b_i \in C_1(R_1)$, $\varphi \in C_2[0,1]$, $\psi \in C_1[0,1]$;

2) h_0 has only isolated and simple zeros on R_1;

3) h_1 has no zeros on R_1.

Let, in the rectangle $\Pi_{T^*} = \{0 \leq x \leq 1,\ 0 \leq t < T^*\}$, there exist a solution $\mathring{u} \in C_2(\Pi_{T^*})$ of problem (1)–(4), and

$$1)\quad \sup_{\Pi_{T^*}} |\mathring{u}| < +\infty; \tag{6}$$

2) $h_0(\mathring{u}(0,t)) \neq 0$ for $0 \leq t < T^*$; without loss of generality we shall assume that

$$h_0(\mathring{u}(0,t)) > 0 \quad \text{for } 0 \leq t < T^*; \tag{7}$$

$$3)\quad \inf_{0 \leq t < T^*} h_0(\mathring{u}(0,t)) = 0. \tag{8}$$

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* The inequality $0 < T^* < 1/2$ involves no loss of generality.

** We have in mind the restrictions on the function h_0 given in 2°, and inequality (10).

Inequality (6) means that the "graph of the function $\mathring{u}$" for $(x,t) \in \Pi_{T^*}$ belongs to the domain of definition of the coefficients and free terms of equations (1), (3), (4); however, equality (8) excludes the possibility of applying, for $t \leq T^*$, the uniqueness and continuation theorems for a solution with preservation of smoothness*, established in (2).

Lemma 1. There exists

$$\lim_{t \to T^*} \mathring{u}(0,t) = u^* \tag{9}$$

and $h_0(u^*) = 0$.

By virtue of Lemma 1 and inequality (7), either $\mathring{u}(0,t) < u^*$ or $\mathring{u}(0,t) > u^*$ for $0 \leq t < T^*$. Without loss of generality we shall assume that $\mathring{u}(0,t) < u^*$.

2. For $(x,t) \in \Pi_{T^*}$ put $\mathring{v} = \mathring{u}_x + \mathring{u}_t$.

Lemma 2. The functions $\mathring{u}, \mathring{v} \in C(\overline{\Pi}_{T^*})^{**}$.

Let

$$\Gamma^0 = f_0(T^*, u^*) - a_0(u^*)\mathring{v}(0,T^*) \neq 0. \tag{10}$$

Denote by $\overline{\Pi}'_{T^*}$ the rectangle $\overline{\Pi}_{T^*}$ with the point $(0,T^*)$ "punctured."

Theorem 1. The function $\mathring{u} \in C_2(\overline{\Pi}'_{T^*})$ and

$$|\mathring{u}_x(x,t)| \to +\infty, \qquad |\mathring{u}_t(x,t)| \to +\infty, \quad \text{if} \ \ t - (x + T^*) \to 0,$$

$$\sup_{\overline{\Pi}'_{T^*}} |\mathring{u}_x(x,t)|\sqrt{|t - (x+T^*)|} < +\infty,$$

$$\sup_{\overline{\Pi}'_{T^*}} |\mathring{u}_t(x,t)|\sqrt{|t - (x+T^*)|} < +\infty.$$

The properties of the derivatives $\mathring{u}_x, \mathring{u}_t$ stated in Theorem 1 are determined by the properties of the function h_0, and not by the smoothness and compatibility of the data of problem (1)–(4). Theorem 1 holds even in the case of analytic and arbitrarily well compatible data.

3°. Discontinuous solution of problem (1)–(4).

1. For $T^* < T \leq \frac{1}{2}$ put $K_T^1 = \{0 \leq x \leq t - T^*,\ T^* \leq t \leq T\}$, $K_T^0 = \overline{\Pi}_T \setminus K_T^1$. Let $\mathfrak{A}_T$, $T^* < T \leq \frac{1}{2}$, be the set of functions $u(x,t)$ such that $u = \mathring{u}$ in Π_{T^*}, $u \in C(K_T^0)$, $u \in C_2(K_T^0)$, and

$$\sup_{K_T^0} |u_x(x,t)| \sqrt{|t-(x+T^*)|} < +\infty,$$

$$\sup_{K_T^0} |u_t(x,t)| \sqrt{|t-(x+T^*)|} < +\infty.$$

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[*] It can be shown that if $\inf_{0 \le t < T^*} h_0(\mathring{u}(0,t)) \neq 0$, then in $\overline{\Pi}_{T^*+\Delta T}$, for some $\Delta T > 0$, there exists a unique solution of problem (1)–(4) of class $C_2(\overline{\Pi}_{T^*+\Delta T})$.

[**] That is, the functions $\mathring{u}, \mathring{v}$ admit a continuous extension as $t \to T^*$.

Definition 1. A function $u \in \mathfrak{R}_T$, $T^* < T \le 1/2$, is called a **solution of problem (1)–(4) in** K_T^0 if u satisfies equation (1) in K_T^0 and equation (4) for $T^* \le t \le T$.

Lemma 3. The following alternative holds:

A. The solution u of problem (1)–(4) exists and is unique in $K_{1/2}^0$.

B. There exists a $T^* < T_0 \le 1/2$ such that the solution u exists and is unique in $K_{T_0-\varepsilon}^0$ for any $\varepsilon > 0$, and

$$\max_{K_{T_0-\varepsilon}^0} |\breve{u}| + \sup_{K_{T_0-\varepsilon}^0} |\breve{u}_x(x,t)| \sqrt{|t-(x+T^*)|} + \sup_{K_{T_0-\varepsilon}^0} |\breve{u}_t(x,t)| \sqrt{|t-(x+T^*)|} \to +\infty \quad \text{as } \varepsilon \to 0.$$

Put $T_1 = 1/2$ in case A and $T_1 = T_0 - \varepsilon_0$, where $0 < \varepsilon_0 < T_0$ is any fixed number, in case B.

Lemma 4. The function $\breve{v} = \breve{u}_x + \breve{u}_t \in C(\breve{K}_{T_1}^0)$.

Below the functions $\breve{u}, \breve{v}$ are denoted as before by $\dot{u}, \dot{v}$.

2. Let

$$H_0(u) = \int_{\varphi(0)}^{u} h_0(\xi)\, d\xi, \qquad I^* = \int_0^{T^*} \left[f_0(\tau, \dot{u}(0,\tau)) - a_0(\dot{u}(0,\tau))\dot{u}(0,\tau) \right] d\tau.$$

Lemma 5. The value $u = \bar{u}^*$ is a root of the equation

$$H_0(u) = I^*. \tag{11}$$

Suppose that equation (11) has a real root $u^* < \bar{u}^* < +\infty$ such that

$$h_0(\bar{u}^*) > 0; \qquad H_0(u) \neq I^* \quad \text{for } u^* < u < \bar{u}^*.$$

Denote by $\mathfrak{N}_T,\ T^* < T \leq T_1$, the set of functions $u \in C_2(K^1_T)$ satisfying the condition $u(0, T^*) = \bar{u}^*$.

Definition 2. A function $u \in \mathfrak{N}_T,\ T^* < T \leq T_1$, is called a **solution of problem (1)–(4) in K^1_T** if:

1) u satisfies equation (1) in K^1_T and equation $(3')$ for $T^* \leq t \leq T$;
2) for $T^* \leq t \leq T$ the relations

$$h_0(u(0,t)) > 0,$$

$$v(t - T^*, t) - P(t - T^*, t, u(t - T^*, t)) =$$

$$= \dot{v}(t - T^*, t) - P(t - T^*, t, \dot{u}(t - T^*, t)),$$

hold, where

$$v = u_x + u_t, \qquad P(x, t, u) = \frac{1}{2}\int_0^u [A(x, t, \eta) - B(x, t, \eta)]\, d\eta.$$

Let the function $u \in \mathfrak{N}_T,\ T^* < T \leq T_1$, be a solution of problem (1)–(4) in K^1_T.

Definition 3. The function $u_p = u$, if $(x, t) \in K^1_T$, and $u_p = \dot{u}$, if $(x, t) \in K^0_T$, is called an r.r. solution of problem (1)–(4) in Π_T.

Theorem 2. The following alternative holds:

1) in $K^1_{T_1}$ there exists a unique solution u of problem (1)–(4), and

$$h_0(u(0,t)) > 0 \quad \text{for } T^* \leq t \leq T_1;$$

2) there exists a $T^* < \widetilde{T} \leq T_1$ such that for any $\varepsilon > 0$ in $K^1_{\widetilde{T}_-}$ there exists a unique solution u of problem (1)–(4), and

$$h_0(u(0,t)) > 0 \quad \text{for } T^* \leq$$

$\leqslant t \leqslant \widetilde{T} - \varepsilon$; moreover:

a) either

$$\sup_{\substack{T^* \leqslant t < \widetilde{T} \\ 0 \leqslant x \leqslant t - T^*}} |u| < +\infty$$

and there exists

$$\lim_{t \to \widetilde{T}} u(0,t) = u^{**}, \qquad h_0(u^{**}) = 0;$$

b) or

$$\sup_{\substack{T^*\leqslant t<\widetilde{T}\\ 0\leqslant x\leqslant t-T^*}} |u| = +\infty.$$

It is not difficult to construct examples showing that all the cases indicated in Theorem 2 are in fact realized.

Put $T_2 = T_1$ in case 1) of Theorem 2, and $T_2 = \widetilde{T} - \varepsilon_0$, where $0 < \varepsilon_0 < \widetilde{T}$ is any fixed number, in case 2).

Theorem 3. In $\overline{\Pi}_{T_2}$ there exists a unique d.r. u_p of problem (1)–(4), and for $T^* \leqslant t \leqslant T_2$

$$u(t-T^*,t) - \dot{u}(t-T^*,t) = \overline{u}^* - u^* +$$

$$+\int_{T^*}^{t} [P(\tau-T^*,\tau,u(\tau-T^*,\tau)) - P(\tau-T^*,\tau,\dot{u}(\tau-T^*,\tau))]\, d\tau.$$

4°. Small perturbations of equation (3) and a discontinuous solution. Following A. A. Vitt (4), consider in $\overline{\Pi}_{T_2}$ the mixed problem (1′)–(4′), defined by equations (1), (2), (4) and the boundary condition

$$\mu u_{tt} + a_c(u)u_x + b_0(u)u_t = f_0(t,u) \qquad \text{for } x = 0. \tag{3'}$$

We note that the functions φ, ψ, generally speaking, do not satisfy the compatibility condition of problem (1′)–(4′) for $x = 0$. Consequently, for any $T > 0$ in $\overline{\Pi}_T$ there is no solution of problem (1′)–(4′) of class $C_2(\Pi_T)$.

For $0 < T \leqslant T_2$ put $G_T^0 = \{(x,t),\ 0 \leqslant x \leqslant T\}$, $G_T^1 = \overline{\Pi}_T \setminus G_T^0$. We now introduce the following

Definition 4. A function $u_\mu(x,t)$ is called a solution of problem (1′)–(4′) in $\overline{\Pi}_T$, $0 < T \leqslant T_2$, if $u_\mu \in C_1(\overline{\Pi}_T)$, $u_\mu \in C_2(\overset{i}{\overline{G}}_T)$ $(i = 0,1)$, and u_μ satisfies equation (1) in the domains $\overset{i}{\overline{G}}_T$ and equations (2), (3′), (4) for $0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T$.

Theorem 4. One can specify a $\mu^* > 0$ such that, for $0 < \mu < \mu^*$, in $\overline{\Pi}_{T_2}$ there exists a unique solution u_μ of problem (1′)–(4′), and

$$\max_{\overline{\Pi}_{T_2}} |u_\mu| < C < +\infty$$

for $0 < \mu < \mu^*$, where C is a constant independent of μ.

Theorem 5. Whatever closed domain $\Omega \subseteq \overline{\Pi}_{T_2}$ not having a common point with the characteristic $t = x + T^*$ may be,

$$\max_{\Omega} |u_\mu - u_p| + \max_{\Omega} |\partial u_\mu/\partial x - \partial u_p/\partial x| + \max_{\Omega} |\partial u_\mu/\partial t - \partial u_p/\partial t| \to 0$$

as $\mu \to 0$.

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Received
28 II 1968

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Note: Figure translations are in progress. See original paper for figures.