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Abstract

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MATHEMATICS

Yu. V. KOMLENKO, E. L. TONKOV

A PERIODIC BOUNDARY-VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER

(Presented by Academician V. I. Smirnov, May 3, 1967)

Consider the periodic boundary-value problem

$$\begin{aligned} x''(t) + q(t)x'(t) + p(t)x(t) &= f(t), & t \in [0, \omega], \\ x(0) = x(\omega), & \quad x'(0) = x'(\omega), \end{aligned} \quad (1)$$

under the assumption that $q(t)$, $p(t)$, and $f(t)$ are real functions of the real variable t , summable on $[0, \omega]$. By $G(t, s)$ we denote the Green's function of problem (1), i.e., such a function that the solution $u(t)$ of problem (1) can be written in the form $u(t) = \int_0^\omega G(t, s)f(s) ds$ for any function $f(t)$. The question of the existence of the Green's function $G(t, s)$ (the case of unique solvability of (1)) and of the preservation of the sign of $G(t, s)$ (the case of applicability of the theorem on a differential inequality) is of great interest⁽¹⁻⁴⁾. Below we propose a number of solutions to the questions mentioned. In what follows, all quantities, unless otherwise specified, are assumed to be real and finite.

1. Let $\alpha = r_0(\alpha), r_1(\alpha), r_2(\alpha), \dots$ be the consecutive zeros of some nontrivial solution of the equation

$$x''(t) + q(t)x'(t) + p(t)x(t) = 0 \quad t \in (-\infty, \infty), \quad (2)$$

where $q(t)$, $p(t)$ are summable ω -periodic functions. Then, by the well-known Sturm theorem, any nontrivial solution of equation (2) has exactly n zeros on $[\alpha, r_n(\alpha))$. The following assertion holds (cf. (5)):

Lemma 1. *If, for every $\alpha \in [0, \omega)$ and some $n = 0, 1, 2, \dots$, the inequalities*

$$r_{2n}(\alpha) < \alpha + \omega < r_{2n+2}(\alpha) < \infty, \quad (3)$$

are satisfied, then the Green's function of problem (1) exists.

Lemma 1 makes it possible to formulate effective criteria for the unique solvability of problem (1) on the basis of sufficient conditions guaranteeing inequalities (3). Such conditions may be found, for example, in (6,7). Denote $\varphi_+ = 2^{-1}(|\varphi| + \varphi)$. For the equation

$$x''(t) + p(t)x(t) = f(t), \quad t \in (-\infty, \infty) \quad (4)$$

we give two criteria.

Theorem 1. *If, for some $n = 0, 1, 2, \dots$ and some $k \in [4n^2\pi^2\omega^{-2}, 4(n+1)^2\pi^2\omega^{-2}]$, the summable ω -periodic function $p(t)$, not equivalent to a constant, satisfies the inequalities*

$$p(t) \geq 4n^2\pi^2\omega^{-2}, \quad \int_0^\omega [p(t) - k]_+ dt < 4(n+1)\sqrt{k} \operatorname{ctg} \frac{\omega\sqrt{k}}{4(n+1)} \quad (5)$$

(the first inequality is understood almost everywhere), then, whatever the summable ω -periodic function $f(t)$, equation (4) has a unique continuous periodic solution of period ω .

We note that Theorem 1 is a more precise result than the assertion of Theorem 8 of paper (3) as applied to equation (4).

Theorem 2. *If a summable ω -periodic function $p(t)$, not equivalent to a constant, is such that*

$$\int_0^\omega p(t) dt \geq 0$$

and, for some

$$k \leq 4\pi^2\omega^{-2},$$

the inequality

$$\int_0^\omega [p(t) - k]_+ dt \leq \begin{cases} 4\sqrt{k} \operatorname{ctg} 4^{-1}\sqrt{k}\omega, & \text{for } k > 0, \\ 16\omega^{-1}, & \text{for } k = 0, \\ 4\sqrt{-k} \operatorname{cth} 4^{-1}\sqrt{-k}\omega, & \text{for } k < 0, \end{cases} \quad (6)$$

is satisfied, then, whatever the summable ω -periodic function $f(t)$, equation (4) has a unique continuous periodic solution of period ω .

Theorem 2 refines Corollary 6 of paper (3), where, instead of condition (6), the inequality

$$\int_0^\omega |p(t)| dt \leq 16\omega^{-1}$$

is required.

The estimates in the theorems given cannot be improved. Namely, if the right-hand sides of inequalities (5), (6) are changed by $\varepsilon > 0$, then there will be found

functions $p(t)$ and $f(t)$ such that the weakened inequalities will be satisfied, while equation (4) will have no ω -periodic solution.

2. As examples show, the conditions of Lemma 1 and Theorems 1 and 2 do not guarantee preservation of the sign of the Green' s function of problem (1).

Lemma 2. Suppose the Green' s function $G(t, s)$ of problem (1) exists. Then the condition

$$\omega < r_1(0)$$

is necessary, and the condition

$$\alpha + \omega < r_1(\alpha)$$

for every $\alpha \in [0, \omega)$ is sufficient, in order that the function $G(t, s)$ vanish nowhere at any point of the square $t, s \in [0, \omega]$.

We note that when the sufficient condition of Lemma 2 is fulfilled, the solution of problem (1), if it exists, depends monotonically on the right-hand side $f(t)$, i.e. the theorem on differential inequalities ⁽¹⁾ is valid.

The following lemma refines Theorem 1 of paper ⁽⁸⁾.

Lemma 3. If

$$\alpha + \omega < r_1(\alpha) < \infty$$

for every $\alpha \in [0, \omega)$, then the Green' s function $G(t, s)$ of problem (1) exists and $G(t, s) > 0$ for $t, s \in [0, \omega]$.

Theorem 3. If a function $p(t)$, summable on $[0, \omega]$ and not equivalent to a constant, is such that

$$\int_0^\omega p(t) dt \geq 0$$

and, for some

$$k \leq \pi^2 \omega^{-2},$$

the inequality

$$\int_0^\omega [p(t) - k]_+ dt \leq \begin{cases} 2\sqrt{k} \operatorname{ctg} 2^{-1}\sqrt{k}\omega, & \text{for } k > 0, \\ 4\omega^{-1}, & \text{for } k = 0, \\ 2\sqrt{-k} \operatorname{cth} 2^{-1}\sqrt{-k}\omega, & \text{for } k < 0, \end{cases}$$

is satisfied, then the Green' s function $G(t, s)$ of the problem

$$x'' + p(t)x = 0, \quad t \in [0, \omega], \quad x(0) = x(\omega), \quad x'(0) = x'(\omega)$$

exists and $G(t, s) > 0$, $t, s \in [0, \omega]$.

The estimates of Theorem 3 are unimprovable in the sense of the remark to Theorems 1 and 2.

3. The availability of sufficient information on the linear problem (1) makes it possible, by known methods (see, for example, ⁽²⁻⁴⁾), to obtain existence theorems for a nonlinear equation.

Consider the problem

$$x''(t) = f(t, x(t)), \quad t \in [0, \omega], \quad x(0) = x(\omega), \quad x'(0) = x'(\omega), \quad (7)$$

where the function $f(t, x)$ is defined in the domain $D = \{0 \leq t \leq \omega, |x| < c \leq \infty\}$ and satisfies the Carathéodory conditions ⁽⁹⁾ in this domain.

Theorem 4. Suppose that the function $f(t, x)$ in the domain D ($c = \infty$) satisfies the inequality

$$|f(t, x) + p(t)x| \leq \bar{p}(t)x + r(t),$$

where $p(t)$ is some function satisfying all the requirements of Theorem 3, $\bar{p}(t)$, $r(t) \geq 0$ are functions summable on $[0, \omega]$, with $\bar{p}(t)$ not equivalent to $p(t)$ and

$$\int_0^\omega \bar{p}(t) dt \leq \int_0^\omega p(t) dt.$$

Then there exists at least one continuous solution of problem (7).

The growth conditions on the function $f(t, x)$ in Theorem 4 are weaker than in ⁽³⁾.

4. On the basis of the ideas of ⁽¹⁾ one can obtain theorems on differential inequalities and uniqueness theorems for problem (7).

Following N. Azbelev ⁽¹⁰⁾, we shall say that a function $f(t, x)$ satisfies condition L_1 (L_2) if $f(t, x)$ has the representation

$$f(t, x) = L_1(t, x) - p_1(t)x$$

$$(f(t, x) = L_2(t, x) - p_2(t)x),$$

where the function $L_1(t, x)$ is increasing ($L_2(t, x)$ is decreasing) with respect to x for almost all t .

Theorem 5. Let the function $f(t, x)$ in the domain D satisfy condition L_1 with coefficient $p_1(t)$, satisfying all the requirements of Theorem 3. Suppose, further, that in the domain D there exists a pair of functions $z_1(t)$, $z_2(t)$, $z_1(t) \geq z_2(t)$, with absolutely continuous derivatives $z_1'(t)$, $z_2'(t)$, such that

$$z_1''(t) \geq f(t, z_1(t)), \quad z_2''(t) \leq f(t, z_2(t))$$

almost everywhere on $[0, \omega]$, and moreover

$$z_i(0) = z_i(\omega), \quad z_i'(0) = z_i'(\omega), \quad i = 1, 2.$$

Then there exists at least one continuous solution of problem (7), and for all solutions the estimate

$$z_1(t) \geq u(t) \geq z_2(t), \quad t \in [0, \omega],$$

holds.

Theorem 6. Let the conditions of Theorem 5 be fulfilled. Suppose, in addition, that the function $f(t, x)$ in the domain D satisfies condition L_2 with coefficient $p_2(t)$, satisfying all the requirements of Theorem 3. Then there exists a unique continuous solution $u(t)$ of problem (7), and the estimate

$$z_1(t) \geq u(t) \geq z_2(t), \quad t \in [0, \omega],$$

holds.

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