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MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.946

MATHEMATICS

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ON AN ESTIMATE OF THE SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A PARABOLIC EQUATION OF SECOND ORDER

(Presented by Academician I. G. Petrovskii, 14 VI 1967)

In this paper an upper estimate is obtained for the modulus of the classical solution $u(x,t)$ of the first boundary-value problem with zero conditions on the lateral boundary for the linear parabolic equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j} + a_i(x,t)u\right) - \frac{\partial u}{\partial t} = 0, \tag{1}$$

considered in an expanding domain G of the $(n+1)$-dimensional space of variables $(x,t) = (x_1,\dots,x_n,t)$. The estimate involves the measure of the domain G and a constant depending only on the dimension of the space and on the ellipticity constant of equation (1). Therefore the result is automatically carried over to quasilinear parabolic equations of second order of the corresponding form. A natural restriction on the coefficients of the lower-order terms of equation (1) is given below (condition (A2)).

1. Notation, assumptions. Let E_n be the n-dimensional Euclidean space of the variables $x = (x_1,\dots,x_n)$. Put

$$\Pi_T = \{(x,t) \mid x \in E_n,\ 0 < t < T\}.$$

Let G_T be a domain lying in the layer Π_T $(T > 0)$, and let $\sigma = \mu_{n+1}G_T$ be the $(n+1)$-dimensional Lebesgue measure of the domain G_T. We denote the lateral boundary of the domain G_T by the symbol $\dot{G}_T$. Put

$$D_\tau = \overline{G}_T \cap \{t = \tau\}.$$

It is assumed that for all $\tau \in [0,T]$ the set $D_\tau \neq \emptyset$ and that its diameter is bounded by a constant depending only on τ.

In the domain G_T, together with equation (1), we consider the following boundary conditions

$$u(x,0) = f(x), \qquad (x,0) \in D_0; \tag{2}$$

$$u(x,t) = f_1(x,t), \qquad (x,t) \in \dot{G}_T. \tag{3}$$

A classical solution $u(x,t)$ of problem (1)–(3) is assumed *a priori* to exist; therefore below we state only those conditions which are directly used in deriving the estimate. The solution of problem (1)–(3) has in G_T continuous partial derivatives with respect to the variables x up to order n inclusive. We shall indicate how to weaken this strong restriction on the smoothness of the solution $u(x,t)$. For brevity put

$$w = \max_{(x,T)\in D_T} |u(x,T)|, \qquad W = \max_{(x,0)\in D_0} |u(x,0)|.$$

The coefficients of equation (1) have the necessary smoothness and satisfy in G_T the conditions

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \geq \nu_1 \sum_{i=1}^{n} \eta_i^2, \tag{A1}$$

* Existence theorems for classical and generalized solutions of linear parabolic equations of second order can be found, for example, in [6,10,11].

where $\nu_1 > 0$ is a certain constant,

$$\sum_{i=1}^{n} \frac{\partial a_i(x,t)}{\partial x_i} \leqslant 0. \tag{A2}$$

The symbols $c, c_1, c_2, \ldots$ denote constants depending only on n and ν_1.

2. Main result. The following theorem holds.

Theorem. Suppose that conditions (A1) and (A2) are satisfied; suppose

$$\sigma < T^{n/2+1}/M, \tag{4}$$

where $M = M(n,\nu_1) > 0$ is a certain constant; and suppose that $u(x,t)$ is a solution of problem (1)–(3) with $f_1(x,t) \equiv 0$. Then there exists a constant $M_1 > 0$, depending only on M, such that

$$w < 2^{-T^2/n+1/M_1\sigma^{2/n}} W. \tag{5}$$

First of all let us note that estimate (5) is sufficiently sharp in the sense that the function

$$v(x,t) = \sin\frac{\pi x_1}{\varepsilon} \cdots \sin\frac{\pi x_n}{\varepsilon} \cdot 2^{-t^2/n+1/K\sigma^{2/n}}$$

is a solution of the first boundary-value problem for the heat equation

$$\nu_1 \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} - \frac{\partial v}{\partial t} = 0$$

in the domain $G^* = \{(x,t) \mid t > 0,\ 0 < x_i < \varepsilon,\ i = 1, \dots, n\}$. Here $\sigma = \varepsilon^n t$ is the $(n+1)$-dimensional measure of the domain G_t^*, and $K = (\nu_1 n \pi^2 \log_2 e)^{-1}$.

Estimate (5), valid for expanding domains G_T (the rate of expansion is limited by condition (4)), has been obtained under the restriction (A2) on the coefficients of the lower-order terms. In his dissertation ([8]) and in ([9]), F. O. Porper proved a theorem on stabilization to zero as $t \to +\infty$ of the solution of the Cauchy problem for equation (1), under a condition of the type (A2) and certain additional assumptions on the coefficients $a_i(x,t)$.

Estimate (5) was first obtained in ([4]) for a classical solution of the first boundary-value problem with zero conditions on the lateral boundary for a linear non-self-adjoint parabolic equation of second order. The constant M_1 in that estimate depended on n, the ellipticity constant, and the constant bounding the coefficients of the non-self-adjoint equation and their derivatives.

3. Proof of the theorem is analogous to the proof of Theorem 2.3.1 of ([5]) and relies essentially on the following lemma:

Lemma. Suppose that conditions (A1) and (A2) are satisfied; suppose

$$\sigma < T^{n/2+1}/M_0, \tag{6}$$

where $M_0 = M_0(n, \nu_1) > 0$ is a certain constant; and suppose that $u(x,t)$ is a sign-preserving solution of problem (1)–(3) with $f_1(x,t) \equiv 0$. Then

$$2w < W. \tag{7}$$

We shall not write out the explicit expression of the constant M_0 in terms of n and ν_1; the constants M and M_1 are expressed through M_0 as follows:

$$M = 4^{n/2+1} M_0, \qquad M_1 = 4^{2/n+1} M.$$

In the proof of the lemma a number of results contained in the dissertation of E. M. Landis ([2]) are used. The proof of the lemma of the present note is simpler than the proof of the analogous lemma for a self-adjoint elliptic equation of second order ([7], §§ 7, 8), since it makes it possible to avoid the use of the "macaroni" lemma of M. L. Gerver and E. M. Landis ([7], § 8).

We give a brief exposition of the proof of the lemma. Introduce the following notation:

$$\Pi_{\xi_1\xi_2} = \Pi_T \cap \{\xi_1 < t < \xi_2\}, \qquad 0 < \xi_1 \le T/8, \quad 7T/8 \le \xi_2 < T,$$

$$l^\rho = \{(x,t) \mid (x,t) \in G_T,\ u(x,t) = \rho\},$$

$$G^\rho = \{(x,t) \mid (x,t) \in G_T,\ u(x,t) > \rho\}, \qquad \rho \in [w/2, w],$$

$$G^\rho_{\xi_1\xi_2} = G^\rho \cap \Pi_{\xi_1\xi_2}, \qquad l^\rho_{\xi_1\xi_2} = l^\rho \cap \Pi_{\xi_1\xi_2}, \qquad D^\rho_\tau = G^\rho \cap \{t = \tau\}, \qquad l^\rho_\tau = l^\rho \cap \{t = \tau\},$$

$$|\nabla_x u| = \left(\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{1/2}, \qquad \mu_{n-1} l^\rho_t = \varphi(t,\rho), \qquad \mu_n D_t = \psi(t),$$

$$\mu_{n-1} l^\rho_t$$

denotes the $(n-1)$-dimensional Hausdorff measure of the set l^ρ_t, and $\mu_n D_t$ the n-dimensional Lebesgue measure of the set D_t.

By the maximum principle, for all $0 \le \rho \le w/2$ and all $0 \le \tau \le T$, l^ρ_τ is a nonempty closed set.

By the theorem of A. S. Kronrod and E. M. Landis (1), the image of the set

$$Q_t = \{(x,t) \mid (x,t) \in D_t,\ |\nabla_{xu}| = 0\}$$

under the mapping $u(x,t)$ has linear measure zero, i.e. $\mu_1 u(Q_t, t) = 0$. Consequently, for almost all $\rho \in [w/2, w]$, l^ρ_t contains no points at which $\nabla_{xu} = 0$. Denote this set of full measure by E_t, so that $E_t \subset [w/2, w]$ and $\mu_1 E_t = w/2$.

There exists a subset $A_t \subset E_t$ such that for all $\rho \in A_t$

$$\int_{l^\rho t} |\nabla_{xu}|\, d\theta > \frac{c_1 \varphi(t,\rho)}{(\psi(t))^{1/n}}\, w. \tag{8}$$

The proof of formula (8) is based essentially on the above-mentioned theorem of A. S. Kronrod and E. M. Landis.

If $B^\rho = \{t \mid t \in [0,T],\ \rho \in A_t\}$, then there exists a ρ_0 such that $\mu_1 B^{\rho_0}$ is sufficiently large. Put $B^{\rho_0} = B$ and $B \cap \Pi_{\xi_1\xi_2} = B(\xi_1,\xi_2)$. Using (8), we obtain

$$\int_{l^{\rho_0}_{\xi_1\xi_2}} |\nabla_{xu}|\, ds =$$

$$= \int_{\xi_1}^{\xi_2} \left(\int_{l^{\rho_0} t} |\nabla_{xu}|\, d\theta \right) dt > \int_{B(\xi_1,\xi_2)} \left(\int_{l^{\rho_0} t} |\nabla_{xu}|\, d\theta \right) dt > \frac{wc_2 T^{1/n}}{\sigma^{1/n}} \int_{B(\xi_1,\xi_2)} \varphi(\rho_0,t)\, dt. \tag{9}$$

Apply to equation (1) the $(n+1)$-dimensional Ostrogradsky formula:

$$0 = \int_{G^{\rho_0}_{\xi_1\xi_2}} \left(\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u}{\partial x_j} + a_i(x,t) u \right) \right) dt\, dx \geq$$

$$\geq \nu_1 \int_{l^{\rho_0}_{\xi_1\xi_2}} |\nabla_{xu}|\, ds - n\rho_0 \int_{G^{\rho_0}_{\xi_1\xi_2}} \sum_{i=1}^{n} \frac{\partial a_i(x,t)}{\partial x_i}\, dt\, dx - \int_{D^{\rho_0}_{\xi_1}} (u - \rho_0)\, dx \geq$$

$$\geq \nu_1 \int_{l^{\rho_0}_{\xi_1\xi_2}} |\nabla_{xu}|\, ds - \int_{D^{\rho_0}_{\xi_1}} (u - \rho_0)\, dx. \tag{10}$$

If, for $\xi_1 = T/8,\ \xi_2 = 7T/8$,

$$\int_{B(\xi_1,\xi_2)} \varphi(\rho_0,t)\, dt \geq \frac{T^{(n+1)/2}}{c_3}, \tag{11}$$

then, combining formulas (9) and (10) and applying formula (6), after elementary transformations we obtain

$$0 > \frac{c_4 T^{1/n+n/2+3/2}}{\sigma^{1/n+1}} w - c_5 W > \frac{c_4 T^{1/n+n/2+3/2}}{\left(\frac{T^{n/2+1}}{M_0} \right)^{1/n+1}} w - c_5 W,$$

whence

$$W > wc_6 M_0^{(n+1)/n} \geq 2w,$$

where $M_0 \geq (2c_6^{-1})^{n/(n+1)}$.

If in formula (11) the inequality sign is reversed, then there exist $\xi_1' > T/8,\ \xi_2' < 7T/8,\ \xi_1' \in B,\ \xi_2' \in B$ such that

$$\int_{B(\xi_1', \xi_2')} \varphi(\rho_0, t)\, dt > c_7 T^{1/2} \left(\mu_n D_{\xi_1'}^{\rho_0} + \mu_n D_{\xi_2'}^{\rho_0} \right). \tag{12}$$

Combining (10) and (12), we obtain

$$0 \geq w \frac{c_2 T^{1/n}}{\sigma^{1/n}} \int_{B(\xi_1', \xi_2')} \varphi(\rho_0, t)\, dt - \int_{D_{\xi'}^{\rho_0}} u\, dx >$$

$$> \left(c_8 \frac{T^{1/2+1/n} w}{\sigma^{1/n}} - W \right) \left(\mu_n D_{\xi_1'}^{\rho_0} + \mu_n D_{\xi_2'}^{\rho_0} \right),$$

whence, after cancellation by $\mu_n D_{\xi_1'}^{\rho_0} + \mu_n D_{\xi_2'}^{\rho_0}$, we shall have

$$W > w \frac{c_8 T^{1/2+1/n}}{\sigma^{1/n}} > w \frac{c_8 T^{1/2+1/n}}{\left(\dfrac{T^{n/2+1}}{M_0} \right)^{1/n}} = w c_8 M_0^{1/n} \geq 2w,$$

where $M_0 \geq (2c_8^{-1})^n$. It remains to put

$$M_0 = \max \left[(2c_8^{-1})^n;\ (2c_6^{-1})^{n/(n+1)} \right].$$

The lemma is proved.

4. Remark 1. We relied on the fact of the existence of continuous derivatives with respect to x up to order n inclusive only in one place–in the proof of formula (8). The lemma and, consequently, the theorem can be proved under the usual smoothness requirements if one uses Theorem 8 of paper (3). In this case the proof differs little in idea from the one given, but is technically considerably more complicated.

Remark 2. The lemma and, consequently, the theorem are valid for generalized solutions $u(x,t)$ satisfying equation (1) in the sense of an integral identity, if the coefficients $a_{ij}(x,t)$ and $a_i(x,t)$ are measurable and uniformly bounded in modulus by a positive constant ν_2, and the functions $a_i(x,t)$ can be approximated by their averages in such a way that condition (A2) is satisfied each time for these averages. The scheme of proof of the lemma in this case is the same as in § 7 of paper (7).

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Received
12 VI 1967

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Note: Figure translations are in progress. See original paper for figures.