EMBEDDING OF CERTAIN CLASSES OF FUNCTIONS OF ONE VARIABLE

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Abstract

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EMBEDDING OF CERTAIN CLASSES OF FUNCTIONS OF ONE VARIABLE

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1. Let $\omega(\delta)$ be a nondecreasing function, continuous on $[0, 1]$, satisfying the conditions

$$\omega(0) = 0, \qquad \omega(\delta + \eta) \leq \omega(\delta) + \omega(\eta) \quad \text{for} \quad 0 \leq \delta \leq \eta \leq \delta + \eta \leq 1.$$

Such functions are called **moduli of continuity**.

If a 2π-periodic function $\psi(\theta) \in L^p(0, 2\pi)$ $(1 \leq p \leq \infty)$, then the function

$$\omega_p(\delta, \psi) = \sup_{|h| \leq \delta} \left\{ \int_0^{2\pi} |\psi(\theta + h) - \psi(\theta)|^p \, d\theta \right\}^{1/p} \qquad (0 \leq \delta \leq 1)$$

is called the **modulus of continuity in** L^p of ψ. For the case of nonperiodic functions $f(x) \in L^p(0, 1)$, the modulus of continuity is defined as follows:

$$\omega_p(\delta, f) = \sup_{0 \leq h \leq \delta} \left\{ \int_0^{1-h} |f(x + h) - f(x)|^p \, dx \right\}^{1/p} \qquad (0 \leq \delta \leq 1).$$

If we are given a modulus of continuity $\omega(\delta)$ and a number $p \in [1, \infty]$, then by

$$H_p^\omega \equiv H_p^{\omega(\delta)}$$

we denote the set of all functions $f(x)$ for each of which $\omega_p(\delta, f) = O\{\omega(\delta)\}$.

In 1927 Hardy and Littlewood ([5]) established that if $\psi(\theta) \in \text{Lip}(\alpha, p)$ $(0 < \alpha \leq 1,\ 1 \leq p < \infty)$*, then:

1)
$$\psi \in \text{Lip}\left(\alpha - \frac{1}{p} + \frac{1}{q},\ q\right), \quad \text{when } \alpha p \leq 1, \quad p < q < \frac{p}{1 - \alpha p};$$

2) $\psi(\theta)$ is equivalent to a continuous function $\psi_1(\theta)$, and

$$\psi_1 \in \mathrm{Lip}\left(\alpha - \frac{1}{p} + \frac{1}{q},\ q\right)$$

for all $q > p$, when $\alpha p > 1$.

In the 1930s S. L. Sobolev ([2]) laid the foundations of the general theory of embedding of spaces of functions of several variables. Subsequently this theory developed actively and continues to develop. Here we shall mention only the works of S. M. Nikol' skii ([3,4]), in the first of which a survey of this theory was given, and in the second embedding theorems were established for the classes of functions

$$H_p^{(r_1, r_2, \ldots, r_n)},$$

a special case of which, for functions of one variable and noninteger r_i, reduces to the assertion of Hardy-Littlewood. It seems to us that the main direction of research on embedding theorems concerns the study of properties of functions of several variables, although, as it appears to us, in the case of functions of one variable as well there may be questions and assertions that are not entirely without interest.

In the present paper we shall present several assertions relating to embedding theorems for functions of one variable. In this direction, in connection with the localization of the Marcinkiewicz criterion, we have estab-

* $\psi(\theta) \in \mathrm{Lip}(\alpha, p)$ means that $\omega_p(\delta, \psi) = O(\delta^\alpha)$.

was shown in 1952 that if

$$\sum_{n=1}^{\infty} \frac{1}{n}\,\omega_1\left(\frac{1}{n}, \psi\right) < \infty, \quad \text{then} \quad \int_0^{2\pi} |\psi(\theta)| \ln^+ |\psi(\theta)|\, d\theta < \infty,$$

i.e. the function $\psi \in L \ln^+ L$ (see ([11]), p. 522). In the same work (see p. 523) it is in fact proved that if

$$\sum_{n=1}^{\infty} \frac{1}{n}\,\omega_2^2\left(\frac{1}{n}, \psi\right) < \infty, \quad \text{then} \quad \psi \in L^2 \ln^+ L.$$

Below we shall see (see Theorems 4 and 6) that these assertions are, in a certain sense, unimprovable.

Several results concerning necessary and sufficient conditions (on the function $\omega(\delta)$) for the embedding of classes of functions $H_p^{\omega(\delta)} \subset L_2(0,1)$ and $H_p^{\omega(\delta)} \subset C(0,1)$ $(1 \le p < \infty)$ were established by us recently in the papers ([12], [13]).

The assertions given below are close in type to the result of Hardy–Littlewood mentioned above, and also to the theorem of Konopkov–Stechkin (see ([9]), p. 56), from which, in particular, it follows that if

$$\sum_{n=1}^{\infty} n^{-1+1/p-1/\nu}\,\omega_p\left(\frac{1}{n},\psi\right) < \infty \quad (1 \le p < \nu \le \infty), \quad \text{then} \quad \psi \in L^{\nu}.$$

2. By $\varphi(t)$ we shall denote even, nonnegative, nondecreasing functions on $[0,\infty)$.

Theorem 1. If $f(x) \in L(0,1)$ and

$$\sum_{n=1}^{\infty}[\varphi(n+1)-\varphi(n)]\,\omega_1\left(\frac{1}{n},f\right) < \infty,$$

then

$$\int_0^1 |f(x)|\varphi(|f(x)|)\,dx < \infty, \quad \text{i.e.} \quad f \in L\varphi(L).$$

From Theorem 1 there immediately follow:

Corollary 1. If $f(t) \in L(0,1)$ and

$$\sum_{n=5}^{\infty} \frac{(\ln n)^{\alpha-1}(\ln\ln n)^{\beta}}{n}\,\omega_1\left(\frac{1}{n},f\right) < \infty,$$

then $f \in L(\ln^+ L)^{\alpha}(\ln^+ \ln L)^{\beta}$ for any $\alpha > 0$ and $\beta \in (-\infty,\infty)$.

For $\alpha = 1,\ \beta = 0$ we obtain the result mentioned above (see ([11])).

Corollary 2. If $f \in L(0,1)$, $\beta > 0$, and

$$\sum_{n=5}^{\infty} \frac{(\ln\ln n)^{\beta-1}}{n\ln n}\,\omega_1\left(\frac{1}{n},f\right) < \infty,$$

then $f \in L(\ln^+ \ln L)^{\beta}$.

Theorem 2. If $f \in L(0,1)$ and

$$\sum_{n=1}^{\infty} 2^{-n}\varphi(2^{n+5}\omega_1(2^{-n},f)) < \infty,$$

then $f \in \varphi(L)$.

From this assertion there follows (the case $\varphi(t) = |t|^{\nu}$):

Corollary 3. If $f \in L(0,1)$, the number $\nu > 1$, and

$$\sum_{n=1}^{\infty} n^{\nu-2}\omega_1^{\nu}\left(\frac{1}{n}, f\right) < \infty,$$

then $f \in L^{\nu}(0,1)$.

For the case $\nu = 2$ this result was established by us in ([13]).

Theorem 3. If $f(x) \in L^p(0,1)$, where $1 < p < \nu < \infty$, and

$$\sum_{n=1}^{\infty} n^{\nu/p-2}\omega_p^{\nu}\left(\frac{1}{n}, f\right) < \infty,$$

then $f \in L^{\nu}(0,1)$.

It is not difficult to see (see Corollary 3) that Theorem 3 remains valid also for $p = 1$.

Theorem 3 is a strengthening of the above-mentioned result of Konyushkov-Stechkin and brings it to a final form (see below).

3. In this section we shall give assertions which show that the theorems of Section 2 are, in a certain sense, final. Let us note that Theorem 1 expresses the true essence of the matter for slowly increasing functions $\varphi(t)$, and Theorem 2 for rapidly increasing functions $\varphi(t)$ of power type, since precisely for such functions these theorems are final. For example, the following are true.

Theorem 4. In order that the embedding

$$H_1^{\omega(\delta)} \subset L \ln^+ L$$

hold, it is necessary and sufficient that the inequality

$$\sum_{n=1}^{\infty} \frac{1}{n}\,\omega\left(\frac{1}{n}\right) < \infty$$

be satisfied.

The sufficiency of this assertion follows from Theorem 1 with $\varphi(t) = \ln(1+|t|)$.

Theorem 5. In order that the embedding

$$H_1^{\omega(\delta)} \subset L^{\nu} \qquad (1 < \nu < \infty),$$

hold, it is necessary and sufficient that the inequality

$$\sum_{n=1}^{\infty} n^{\nu-2}\omega^{\nu}\left(\frac{1}{n}\right) < \infty.$$

be satisfied.

The sufficiency of this assertion follows from Theorem 2 if one sets $\varphi(t) = |t|^{\nu}$ (see Corollary 3).

Theorem 3 is also, in a certain sense, final.

We give one more result of analogous type.

Theorem 6. In order that the embedding

$$H_2^{\omega(\delta)} \subset L^2 \ln^+ L$$

hold, it is necessary and sufficient that the inequality

$$\sum_{n=1}^{\infty} \frac{1}{n}\omega^2\left(\frac{1}{n}\right) < \infty.$$

be satisfied.

Let us note that if the 2π-periodic function $\psi(\theta) \in L(0, 2\pi)$ and a_n, b_n are its trigonometric Fourier coefficients, then fulfillment of the inequality

$$\sum_{n=1}^{\infty} \frac{1}{n}\omega_2^2\left(\frac{1}{n}, \psi\right) < \infty$$

is equivalent to fulfillment of the Kolmogorov-Seliverstov condition (7)

$$\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\ln n < \infty.$$

This last fact follows from an assertion of M. K. Potapov (10).

4. In this section we shall make several remarks.

A. In proving the sufficiency of the conditions given above, we used lower estimates of the moduli of continuity $\omega_p(\delta, f)$ in terms of the values of inte–

degrees from the function $|f|$ over those sets E of measure δ, where $\inf_{x\in E}|f(x)| \geq \sup_{x\in[0,1]-E}|f(x)|$.

As for establishing the necessity of the conditions, here we used a certain "regularity" of the behavior of moduli of continuity when properties are considered of series whose terms are powers of the modulus of continuity taken with certain weights.

B. The results given above (see Theorems 2, 3, 5 and Corollary 3) show that, in order to obtain definitive theorems in the case of embedding, for example, into the space L^ν, the condition on $\omega(\delta, f)$ or on $\omega(\delta)$ is expressed through its ν-th power (through $\omega^\nu(\delta, f)$ or $\omega^\nu(\delta)$); i.e., the determining role is played by the space into which the embedding is being considered.

C. The indicated results can be supplemented in the sense that one may require of a function $f \in H_p^\omega$ not only membership in the space L^ν (or $\varphi(L)$), but also that it have in L^ν one or another order of modulus of continuity. One such result was established by us in (13), where the embedding $H_1^{\omega_1(\delta)} \subset H_2^{\omega_2(\delta)}$ was discussed.

D. It is clear that in studying the indicated questions one may consider moduli of smoothness of any order, not only the first.

E. In the paper (12) we indicated a certain connection between some properties of trigonometric series and series with respect to the Haar system and embedding theorems. Here we would like to draw attention to one more fact in the same direction. Namely, from a result of Marcinkiewicz (see (6), the case $p = 1$) it follows directly that if

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega_1\left(\frac{1}{n}, \psi\right) < \infty, \tag{1}$$

then the trigonometric Fourier series $\sigma(\psi)$ converges almost everywhere on $[0, 2\pi]$. In connection with this, Zygmund posed (see (1), p. 258) the question whether the preceding assertion remains true if

$$\omega_1\left(\frac{1}{n}, \omega\right) = O\{(\ln n)^{-1}\}. \tag{2}$$

But from Corollary 1 it follows that condition (1) (respectively (2)) implies

$$\psi \in L \ln^+ L \qquad \left(\psi \in L(\ln^+ L)^{1-\varepsilon} \text{ for every } \ \varepsilon \in (0,1)\right).$$

Therefore, if the answer to Zygmund' s question is negative, then in any case divergent on a set of positive measure trigonometric Fourier series of the class $L(\ln^+ L)^\alpha$ must be constructed for every $\alpha \in (0, 1)$. And this is not known even for a single $\alpha > 0$, although the example of A. N. Kolmogorov (8) of an almost everywhere divergent Fourier series is well known.

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Note: Figure translations are in progress. See original paper for figures.