RINGS WITHOUT NILPOTENT ELEMENTS AND COMPLETELY PRIME IDEALS

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In the present note it is proved that an associative ring is a ring without nonzero nilpotent elements if and only if it is isomorphic to a subdirect sum of rings without zero divisors. This assertion generalizes the analogous well-known result of Krull for commutative rings (see ([1]); ([2]), Theorem 24).

Let K be an arbitrary associative ring. Recall that an ideal $P \neq K$ is called prime if from $AB \subseteq P$, where A and B are ideals in K, it follows that $A \subseteq P$ or $B \subseteq P$. It is known ([3], [4]) that P is a prime ideal if and only if from $aKb \subseteq P$, where a and b are elements of the ring K, it follows that $a \in P$ or $b \in P$. A set M of elements of the ring K is called an m-system if for any $a, b \in M$ there exists an $x \in K$ such that $axb \in M$. It is clear that if P is a prime ideal, then the complement $C(P) = K \setminus P$ is an m-system. In some cases the converse assertion is also true. Namely, with the aid of the Kuratowski-Zorn lemma it is easy to show that any m-system M not intersecting an ideal $A \neq K$ is contained in some maximal m-system M^* not intersecting A. The complement $C(M^*)$ will be a prime ideal—the minimal prime ideal containing the ideal A (see ([3], [4])). Indeed, by the Kuratowski-Zorn lemma, among the ideals of the ring K containing A and not intersecting M^*, there exists a maximal ideal P. Let us show that P is a prime ideal. In fact, if D_1 and D_2 are such ideals in K that $D_1 \supset P$ and $D_2 \supset P$, then, by the maximality of P, in K there exist elements d_1 and d_2 such that $d_1 \in D_1 \cap M^*$, $d_2 \in D_2 \cap M^*$. Since M^* is an m-system, there exists an $x \in K$ such that $d_1 x d_2 \in M^*$. But $d_1 x d_2 \in D_1 D_2$. By the primeness of the ideal P, the complement $C(P)$ is an m-system. From the relations $P \supset A$, $P \cap M^* = \emptyset$ we obtain $C(P) \cap A = \emptyset$, $M^* \subseteq C(P)$. By the maximality of M^*, $M^* C(P) = M^*$, and therefore $C(M^*) = P$. It is clear that P is a minimal prime ideal containing A. Indeed, if there existed a prime ideal P' with $P \supset P' \supset A$, then, on the one hand, $C(P') \cap A = \emptyset$, and, on the other hand, $C(P') \cap A \neq \emptyset$, since $C(P') \supset C(P) = M^*$. Recall now that an ideal $P \neq K$ is called **completely prime** if from $ab \in P$ it follows that $a \in P$ or $b \in P$, i.e., if the factor ring K/P contains no zero divisors. An ideal $Q \neq K$ will be called **completely semiprime** if from $a^n \in Q$ it follows that $a \in Q$, i.e., if the factor ring K/Q is a ring without nonzero nilpotent elements. Obviously, a ring without nonzero nilpotent elements is a ring in which zero is a completely semiprime ideal. It is clear that a completely prime ideal is also a completely semiprime ideal. Moreover, the intersection of any set of completely semiprime ideals is a completely semiprime ideal.

Lemma on the existence of completely prime ideals.
If Q is a completely semiprime ideal and B is such an ideal that $B \supset Q$, then there exists a completely prime ideal P such that $P \supset Q$, but $P \not\supset B$.

Proof. Let b be such an element of the ring K that $b \in B$, $b \notin Q$. Then, by the condition of the lemma, the m-system $M = \{b^i\}$, $i = 1, 2, \dots$, does not intersect the ideal Q. Denote by M^* a maximal m-system containing M and not intersecting the ideal Q. By what was said above

complement $C(M^*) = P$ will be a minimal prime ideal containing Q. Since $b \notin C(M^*) = P$, we have $P \not\supset B$. Let us prove that P is a completely prime ideal. For this purpose consider the set M' of all elements of the form $x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$, where r_i are natural numbers and $x_1 x_2 \dots x_n \in M^*$. Clearly, $M^* \subseteq M'$. Note that M' is an m-system. Indeed, if $a, b \in M'$, then

$$a = a_1^{r_1} a_2^{r_2} \dots a_m^{r_m}, \quad a_1 a_2 \dots a_m \in M^*; \quad b = b_1^{s_1} b_2^{s_2} \dots b_n^{s_n}, \quad b_1 b_2 \dots b_n \in M^*.$$

Since M^* is an m-system, there exists an element $x \in K$ such that $a_1 a_2 \dots a_{mxb} 1 b_2 \dots b_n \in M^*$. But then, by the definition of the set M',

$$axb = a_1^{r_1} a_2^{r_2} \dots a_m^{r_m} x b_1^{s_1} b_2^{s_2} \dots b_n^{s_n} \in M'.$$

We shall now show that the m-system M' does not meet the ideal Q. For this, first note that from $xy \in Q$ it follows that $yx \in Q$, and conversely. Indeed, if $xy \in Q$, then $(yx)^2 = y(xy)x \in Q$, and, since Q is a completely semiprime ideal, $yx \in Q$. Similarly, from $yx \in Q$ it follows that $xy \in Q$. Suppose now that $M' \cap Q \neq \emptyset$, and let $c \in M' \cap Q$. Then $c = c_1^{r_1} c_2^{r_2} \dots c_n^{r_n} \in Q$, $c_1 c_2 \dots c_n \in M^*$. If $r_1 > 1$, then, by the remark just made above, $c_1^{r_1-1} c_2^{r_2} \dots c_n^{r_n} c_1 \in Q$, whence $(c_1^{r_1-1} c_2^{r_2} \dots c_n^{r_n})^2 \in Q$, and therefore $c_1^{r_1-1} c_2^{r_2} \dots c_n^{r_n} \in Q$. After a finite number of steps we obtain $c_1 c_2^{r_2} \dots c_n^{r_n} \in Q$. But then $c_2^{r_2} \dots c_n^{r_n} c_1 \in Q$. Repeating the same arguments for $r_2, \dots, r_n$, after a finite number of steps we obtain $c_1 c_2 \dots c_n \in Q$, and therefore $c_1 c_2 \dots c_n \in Q \cap M^*$. But, by the choice of M^*, $Q \cap M^* = \emptyset$. The contradiction obtained proves that $Q \cap M' = \emptyset$. Hence, and from the maximality of the m-system M^*, we obtain $M' = M^* = C(P)$. Note now that from $a \in M^*$ it follows that $a^2 \in M^*$. Indeed, by the definition, $M'a^2 \subseteq M'$. Finally, let us prove that P is a completely prime ideal. Let $ab \in P$. Then $(ba)^2 \in P$, whence $(ba)^2 \notin C(P) = M^*$. Consequently, $ba \notin M^*$, and therefore $ba \in C(M^*) = P$. But then for any $x \in K$, from $ab \in P$ it follows that $abx \in P$, $bxa \in P$, i.e. $bKa \subseteq P$. By the primeness of the ideal P we get $b \in P$ or $a \in P$. The lemma is proved.

Note that, since $M^* = C(P)$ and P is a completely prime ideal, from the proof of the lemma itself we obtain

Corollary 1. *Every maximal m-system M^* not meeting a given completely semiprime ideal $Q \neq K$ is a multiplicative system, i.e. from $a \in M^*$, $b \in M^*$ it follows that $ab \in M^*$.*

Corollary 1. *In every ring $K \neq 0$ without nonzero nilpotent elements there exists a completely prime ideal not containing a given nonzero ideal.*

Indeed, in the assumptions of the lemma it suffices to put $Q = 0$.

Theorem 1. *Every completely semiprime ideal of a ring is the intersection of all completely prime ideals containing it.*

Indeed, let $B = \bigcap_\alpha P_\alpha$, where P_α runs through all completely prime ideals containing the completely prime ideal Q. Clearly, $B \supseteq Q$. Suppose that $B \supset Q$. Then, by the lemma, there exists a completely prime ideal P_{α_0} such that $P_{\alpha_0} \supseteq Q$ and $P_{\alpha_0} \not\supset B$, which contradicts the definition of the ideal B. Consequently,

$$Q = B = \bigcap_\alpha P_\alpha.$$

It follows at once from Theorem 1 that the intersection D_1 of all completely semiprime ideals containing a given ideal A coincides with the intersection D_2 of all completely prime ideals containing A. Indeed, it is clear that $A \subseteq D_1 \subseteq D_2$ and, since D_1 is a completely semiprime ideal, by Theorem 1,

$$D_1 = D_2.$$

Theorem 2. *In order that a ring $K \neq 0$ be a ring without nilpotent elements $\neq 0$, it is necessary and sufficient that K be isomorphic to a subdirect sum of rings without zero divisors (see also* (10)*).*

Indeed, suppose first that K is a subdirect sum of rings without zero divisors. Since any subring of a complete direct sum of rings without zero divisors is, obviously, a ring without nonzero nilpotent elements, K will also be a ring of the same kind. Suppose now that the ring $K \neq 0$ has no nilpotent elements $\neq 0$. Since 0 is a completely prime ideal, by Theorem 1,

$$0 = \bigcap_\alpha P_\alpha,$$

where P_α ranges over all completely prime ideals of the ring K. Consequently, K is isomorphic to a subdirect sum of rings without zero divisors $\overline{K}_\alpha = K/P_\alpha$. Let K be an arbitrary associative ring. We construct by induction a chain of ideals of the ring K

$$0 = \mathfrak{N}_0(K) \subseteq \mathfrak{N}_1(K) \subseteq ... \subseteq \mathfrak{N}_i(K) \subseteq \mathfrak{N}_{i+1}(K) \subseteq ... ,$$

where i ranges over the natural numbers, and $\mathfrak{N}_{i+1}(K)$ is the ideal of the ring K generated by the set

$$\mathfrak{n}_i(K) = \{a \mid a \in K,\ a^n \in \mathfrak{N}_i(K) \text{ for some natural } n = n(a)\},$$

that is, by the set of all elements of the ring K nilpotent modulo $\mathfrak{N}_i(K)$. Put

$$\mathfrak{N}(K) = \bigcup_0^\infty \mathfrak{N}_i(K).$$

Proposition 2. *The ideal $\mathfrak{N}(K)$ is the least completely semiprime ideal of the ring K, i.e. the least among all such ideals Q for which the quotient ring K/Q has no nilpotent elements $\neq 0$.*

Indeed, let Q be a completely semiprime ideal. It is clear that $\mathfrak{N}_0 \subseteq Q$. If it has already been proved that $\mathfrak{N}_i \subseteq Q$ for some i, then $\mathfrak{n}_i(K) \subseteq Q$. But then also $\mathfrak{N}_{i+1} \subseteq Q$. Consequently, for all i, $\mathfrak{N}_i \subseteq Q$, and therefore $\mathfrak{N}(K) \subseteq Q$. On the other hand, $\mathfrak{N}(K)$ is a completely semiprime ideal, since, if $b^n \in \mathfrak{N}(K)$, then $b^n \in \mathfrak{N}_i(K)$ for some i, and therefore $b \in \mathfrak{N}_{i+1}(K) \subseteq \mathfrak{N}(K)$.

From Proposition 2 and Theorem 1 we obtain

Proposition 3. *The ideal $\mathfrak{N}(K)$ is the intersection of all completely prime ideals of the ring.*

Since the generalized nil radical N_g of the ring K also coincides with the intersection of all its completely prime ideals (see ([5],[6]); ([7]), p. 153), it follows that

Corollary 2. *The ideal $\mathfrak{N}(K)$ coincides with the generalized nil radical N_g of the ring K, or, what is the same, with the compressive radical (see ([8])).*

Corollary 3. *The ideal $\mathfrak{N}(K)$ is a special radical. The $\mathfrak{N}$-radical rings are precisely the rings that do not map onto nonzero rings without zero divisors, or, what is the same, the rings without completely prime ideals. The $\mathfrak{N}$-semiprime rings coincide with subdirect sums of rings without zero divisors, i.e. with rings without nonzero nilpotent elements. For any ideal A of the ring K, the equality $\mathfrak{N}(A) = A \cap \mathfrak{N}(K)$ holds.*

From what was said above it follows that the generalized nil radical N_g is the least (see ([9])) among all such radicals r that in any ring K all nilpotent elements belong to the r-radical $r(K)$ of the ring K, i.e. $N_g(K) \subseteq r(K)$ for every such radical. Consider the property ν of being the ring generated as an ideal by its nilpotent elements. Then from the preceding it follows that the generalized nil radical N_g coincides with the lower radical in the sense of Kurosh generated by the property ν (see ([9])). In conclusion we note that the majority of the results obtained carry over without change to semigroups.

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Note: Figure translations are in progress. See original paper for figures.