A PRIORI ESTIMATES OF SOLUTIONS OF ELLIPTIC EQUATIONS IN THE CLASS OF ANALYTIC FUNCTIONS

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Abstract

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A PRIORI ESTIMATES OF SOLUTIONS OF ELLIPTIC EQUATIONS IN THE CLASS OF ANALYTIC FUNCTIONS

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Let u be the solution of the Dirichlet problem for an elliptic equation of second order with analytic coefficients in a domain $G \subset R^n$ with analytic boundary $\dot{G}$:

$$\sum_{|\beta| \le m} a_\beta(x) D^\beta u = f; \qquad x \in G; \qquad u = \varphi; \qquad x \in \dot{G}.$$

The main result of the paper is the establishment of estimates up to the boundary for the solution u and derivatives of the solution in the class of analytic functions. In obtaining these estimates we used the ideas of the work ([1]).

§ 1. **Norms and their properties.** We shall denote by T_j first-order differential operators with analytic coefficients:

$$T_j = \sum_{i=1}^{n} b_{ij}(x) D_i, \qquad 1 \le j \le s.$$

The operators T_j satisfy two assumptions:

1. On the boundary $\dot{G}$ the operators T_j are differentiation operators in directions tangent to the boundary.
2. At every point $x \in \dot{G}$, derivatives in tangential directions can be represented in the form of linear combinations of the T_j.

As such operators one may take the operators $a_j(x)D_i - a_i(x)D_j$, where the functions $a_j(x)$, analytic in G, are equal on the boundary $\dot{G}$ to $\cos(N, x_j)$ (N is the exterior normal to $\dot{G}$).

For functions $f \in C_{\infty,k+\alpha}(G \cup \dot{G})$, i.e. Hölder-continuous with exponent α, together with all their derivatives of the form

$$D^\beta f,\ (D^\beta T_{j_1} \dots T_{j_l}) f; \qquad |\beta| \le k, \qquad l = 1, 2, \dots,$$

we introduce

$$|f|_{0,k+\alpha} = \|f\|_{C_{k+\alpha}(G)}, \qquad |f|_{l,k+\alpha} = \max_{j_\nu} \left\| \left(\prod_{\nu=1}^{l} T_{j_\nu} \right) f \right\|_{C_{k+\alpha}(G)},$$

$$|\dot f|_{0,k+\alpha} = \|f\|_{C_{k+\alpha}(\dot G)}, \qquad |\dot f|_{l,k+\alpha} = \max_{j_\nu} \left\| \left(\prod_{\nu=1}^{l} T_{j_\nu} \right) f \right\|_{C_{k+\alpha}(\dot G)}.$$

The role of norms in $C_{\infty,k+\alpha}$ will be played by the formal power series

$$\|f\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} |f|_{l,k+\alpha}, \qquad \|\dot f\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} |\dot f|_{l,k+\alpha}.$$

For analytic f, the series $\|f\|_{\rho,k+\alpha}$ and $\|\dot f\|_{\rho,k+\alpha}$ converge for sufficiently small ρ. Indeed, since f and b_{ij} are analytic, we have

$$\|D^\beta f\|_{C_{k+\alpha}(G)} \le M_1 R_1^{|\beta|} |\beta|!; \qquad \|D^\beta b_{ij}\|_{C_{k+\alpha}(G)} \le M_2 R_2^{|\beta|} |\beta|!.$$

The convergence of $\|f\|_{\rho,k+\alpha}$ will follow from

$$\left\| D^\beta \left(\prod_{\nu=1}^{l} T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} \ll M R^{|\beta|} \lambda^l (|\beta| + l)! \tag{1}$$

Choose $R > \max(R_1, R_2)$, $\lambda > M_2 R^2 n / (R - R_2)$, $M \ge M_1$. We shall prove the inequality by induction on l. Suppose that it is true for $l \le \sigma - 1$. Then

$$\begin{aligned}
\left\| D^\beta \left(\prod_{\nu=1}^{\sigma} T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} &\ll \sum_{i=1}^{n} \left\| D^\beta \left(b_{ij_1} D_j \left(\prod_{\nu=2}^{\sigma} T_{j_\nu} \right) f \right) \right\|_{C_{R+\alpha}(G)} \\
&\ll \sum_{i=1}^{n} \sum_{p+q=\beta} \binom{\beta}{p} \left\| D^p b_{ij_1} \right\|_{C_{R+\alpha}(G)} \left\| D_i^{qD} \left(\prod_{\nu=2}^{\sigma} T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} \\
&\ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)! \frac{M_2 R^2 n}{\lambda} \sum_{m=0}^{|\beta|} \frac{|\beta|! (|\beta| + \sigma - m)!}{(|\beta| - m)! (|\beta| + \sigma)!} \left(\frac{R_2}{R} \right)^m \\
&\ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)! \frac{M_2 R^2 n}{\lambda (R - R_2)} \ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)!.
\end{aligned}$$

Thus, if inequality (1) is valid for $l = 1$, then it is valid for all l. The case $l = 1$ is verified separately, and the proof of the inequality is carried out in the same way.

Since an analytic function defined on $\dot{G}$ can be analytically continued inward, the convergence of $\|f\|_{\rho,k+\alpha}$ follows from the preceding.

We note that from the convergence of the series $\|f\|_{\rho,k+\alpha}$ for $\rho \le \rho_0$, generally speaking, analyticity of f in G does not follow. Consider, for example, the strip $G = \{x : -\infty < x_1, \dots, x_{n-1} < \infty,\ 0 < x_n < 1\}$. Put $T_j = D_j$, $j = 1, 2, \dots, n-1$. Then from the convergence of the series $\|f\|_{\rho,k+\alpha}$ analyticity of f with respect to $x_1, x_2, \dots, x_{n-1}$ will follow, but not with respect to x_n.

We now formulate the basic properties of the norms in the form of lemmas, the proof of which will be given at the end of the paragraph.

Lemma 1. For functions $f, g \in C_{\infty,k+\alpha}$ the estimates

$$\|T_j f\|_{\rho,k+\alpha} \ll \frac{\partial}{\partial \rho}\|f\|_{\rho,k+\alpha}, \tag{2}$$

$$\|fg\|_{\rho,k+\alpha} \ll \|f\|_{\rho,k+\alpha}\|g\|_{\rho,k+\alpha}. \tag{3}$$

hold. Analogous inequalities, by virtue of assumption 1 on the operators T_j, are also valid for $\|\cdot\|_{\rho,k+\alpha}$.

For convenience, introduce $q_k(\rho)$ and $q(\rho)$ so that

$$\sum_{i=1}^{n} \|D_m b_{ij}\|_{\rho,k+\alpha} \ll q_k(\rho); \qquad 1 \le m \le n, \quad 1 \le j \le s;$$

$$q_k(\rho) \ll q(\rho), \quad 0 \le k \le r-1, \tag{1'}$$

Lemma 2. Let $f \in C_{\infty,k+1+\alpha}$. Then

$$\|D_j f\|_{\rho,k+\alpha} \ll \exp\left\{\int_0^\rho q_k(\xi)\, d\xi\right\} \|f\|_{\rho,k+1+\alpha}. \tag{4}$$

For the differential operator

$$a(x, D) = \sum_{|\beta| \le m} a_\beta(x) D^\beta$$

define

$$\|a\|_{\rho,k+\alpha} = \sum_{|\beta|\le m} \|a_\beta\|_{\rho,k+\alpha}.$$

Denote

$$\|[a]f\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} \max_{j_\nu} \left\| \left[a, \prod_{\nu=1}^{l} T_{j_\nu} \right] f \right\|_{C_{k+\alpha}(G)},$$

where

$$\left[a, \prod_{\nu=1}^{l} T_{j_\nu} \right] = a \prod_{\nu=1}^{l} T_{j_\nu} - \left(\prod_{\nu=1}^{l} T_{j_\nu} \right) a.$$

Lemma 3. If $f \in C_{\infty,k+m+\alpha}$, $a \in C_{\infty,k+\alpha}$, then for $k+m \le r$ we have

$$\|[a]f\|_{\rho,k+\alpha} \ll \exp\left\{ m \int_0^\rho q(\xi)\, d\xi \right\} \left[\|a\|_{\rho,k+\alpha} - \|a\|_{0,k+\alpha} + m\rho q(\rho) \|a\|_{\rho,k+\alpha} \right] \|f\|_{\rho,k+m+\alpha}. \tag{5}$$

The proof of Lemmas 1-3 is not given for lack of space.

§ 2. The Dirichlet problem

Theorem 1. Let $G \subset R^n$ be a bounded domain with analytic boundary $\dot{G}$, and let $L(x,D)$ be a uniformly elliptic differential operator of second order with coefficients from $C_{\infty,k-2+\alpha}$. Then for $u \in C_{\infty,k+\alpha}$ the estimate

$$\|u\|_{\rho,k+\alpha} \ll \frac{c}{1 - c[\rho + A(\rho)]} \left[\|Lu\|_{\rho,k-2+\alpha} + \|\dot{u}\|_{\rho,k+\alpha} + \|u\|_{C_0(G)} \right], \tag{6}$$

holds, where

$$A(\rho) = \exp\left\{ 2 \int_0^\rho q(\xi)\, d\xi \right\} \left[\|L\|_{\rho,k-2+\alpha} - \|L\|_{0,k-2+\alpha} + 2\rho q(\rho) \|L\|_{0,k-2+\alpha} \right].$$

Here $q(\rho)$ is the formal power series from $(1')$ with $r = 2$. The constant c depends on k, α, n, the boundary $\dot{G}$, the ellipticity constant of the operator L, and the norms of the coefficients of the operator L in $C_{k-2+\alpha}(G)$.

The proof is based on the well-known Schauder estimate (2):

$$|v|_{0,k+\alpha} \ll c \left[|Lv|_{0,k-2+\alpha} + |\dot{v}|_{0,0} \right]. \tag{7}$$

Putting $v = \left(\prod_{\nu=1}^{l} T_{j_\nu}\right) u$ and using Proposition 1 for the operators T_j, we obtain the inequality

$$|u|_{l,k+\alpha} \ll C \left[\max_{j_\nu} \left| L \left(\prod_{\nu=1}^{l} T_{j_\nu} \right) u \right|_{0,k-2+\alpha} + |\dot{u}|_{l,k+\alpha} + |u|_{l,0} \right],$$

from which it follows that

$$\|u\|_{\rho,k+\alpha} \ll c \left[\|Lu\|_{\rho,k-2+\alpha} + \|[L]u\|_{\rho,k-2+\alpha} + \|\dot{u}\|_{\rho,k+\alpha} + \|u\|_{\rho,0} \right].$$

Since

$$\|u\|_{\rho,0} \ll |u|_{0,0} + \text{const} \cdot \rho \|u\|_{\rho,1} \ll \|u\|_{C_0(G)} + \text{const} \cdot \rho \|u\|_{\rho,k+\alpha},$$

(5) immediately gives (6).

Theorem 2. Suppose that the hypotheses of Theorem 1 are satisfied. Then

$$\|D_j u\|_{\rho,k-1+\alpha} \ll \frac{c}{1 - c[\rho + A(\rho)]} \exp\left\{ \int_0^\rho q(\xi)\, d\xi \right\} \left[\|Lu\|_{\rho,k-2+\alpha} + \left(1 + \frac{\partial}{\partial\rho}\right) \|u\|_{\rho,k-1+\alpha} \right], \tag{8}$$

where the constant c depends on the parameters indicated in Theorem 1.

Proof. We shall use Proposition 2 for the operators T_j:

$$\|u\|_{C_{k+\alpha}(\dot{G})} \ll \text{const} \cdot \left[\|u\|_{C_{k-1+\alpha}(\dot{G})} + \sum_{j=1}^{s} \|T_j u\|_{C_{k-1+\alpha}(\dot{G})} \right.$$

Consequently,

$$\|\dot{u}\|_{\rho,k+\alpha} \ll \text{const.}\, (1 + \partial/\partial\rho) \|\dot{u}\|_{\rho,k-1+\alpha}. \tag{9}$$

Since

$$\|u\|_{C(G)} \ll \|u\|_{\rho,k-1+\alpha}; \qquad \|\dot{u}\|_{\rho,k-1+\alpha} \ll \|u\|_{\rho,k-1+\alpha},$$

then, combining (4), (6), and (9), we obtain (8).

Remark 1. If the uniqueness theorem for the solution of the Dirichlet problem is valid for the operator L, then instead of (7) one may use the estimate

$$|v|_{0,k+\alpha} \ll c[\|Lv\|_{0,k-2+\alpha} + |v|_{0,k+\alpha}]. \tag{10}$$

Then

$$\|u\|_{\rho,k+\alpha} \ll \frac{c}{1-cA(\rho)}\left[\|Lu\|_{\rho,k-2+\alpha} + \|\dot{u}\|_{\rho,k+\alpha}\right]. \tag{11}$$

Combining (9) and (11), we have

$$\|D_j\dot{u}\|_{\rho,k+\alpha} \ll \left(1+\frac{\partial}{\partial\rho}\right)\left[\frac{c}{1-cA(\rho)}(\|Lu\|_{\rho,k-2+\alpha} + \|\dot{u}\|_{\rho,k+\alpha})\right]. \tag{12}$$

Remark 2. Estimates (6) and (8) are valid for unbounded domains if inequality (7) holds. Conditions sufficient for the existence of (7) are given in ([2]). If the maximum principle is valid in such domains, then inequalities (11) and (12) hold.

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Note: Figure translations are in progress. See original paper for figures.