M. B. BARBAN, B. V. LEVIN

1968

SovietRxiv

Abstract

Full Text

M. B. BARBAN, B. V. LEVIN

MULTIPLICATIVE FUNCTIONS ON “SHIFTED” PRIME NUMBERS

(Presented by Academician I. M. Vinogradov, 29 XI 1967)

Many questions in the theory of prime numbers lead to the necessity of estimating the sum of the values of a multiplicative function $f(m)$ at “shifted” prime numbers. Under certain additional assumptions concerning $f(m)$, the most restrictive of which is the requirement $f(p) \geqslant 1$ (p prime), such estimates were obtained in (^{1}).

However, a number of problems, for example the generalized Hardy–Littlewood problem (^{2}), require for their solution estimates of sums of multiplicative functions which, at prime numbers, take values smaller than 1. For these purposes the following may serve.

Theorem 1. Let $f(m) \geqslant 0$ be a multiplicative function ($f(m_1 m_2) = f(m_1) f(m_2)$ for $(m_1, m_2) = 1$), and suppose that on powers of prime numbers $f(q^\alpha) = O(c^\alpha)$, where c is a constant.

Then

$$\sum_{p \leqslant N} f(N-p) \ll \frac{N}{\ln N} \exp\left\{ \sum_{\substack{q<N \\ (q,N)=1}} \frac{f(q)-1}{q} \right\},$$

where $\ll$ is the symbol of I. M. Vinogradov.

Indeed, the condition $f(q^\alpha) = O(c^\alpha)$ shows that $f(m) = O(m^\varepsilon)$ for any $\varepsilon > 0$. Therefore

$$\sum_{p \leqslant N} f(N-p) \ll N^{3/4} + \frac{1}{\ln^2 N} \sum_{\substack{p<N \\ (p,N)=1}} f(N-p) \ln^2(N-p).$$

But the function $\ln^2 m$ is not only monotone, but also “almost additive”

$$\left(\ln^2 m = \left(\sum_{q^\alpha \| m} \ln q^\alpha \right)^2 \right),$$

which immediately makes it possible to appeal to the estimates of the sieve method:

$$\sum_{\substack{p\leqslant N\\(p,N)=1}} f(N-p)\ln^2(N-p)=$$

$$=\sum_{q_1^{\alpha_1}\leqslant N}\ln q_1^{\alpha_1}\sum_{q_2^{\alpha_2}\leqslant N}\ln q_2^{\alpha_2}\sum_{\substack{p\leqslant N\\ N-p=m[q_1^{\alpha_1},q_2^{\alpha_2}]\\(m,q_1q_2)=1}} f(m)f([q_1^{\alpha_1},q_2^{\alpha_2}])\ll$$

$$\ll\sum_{\substack{m\leqslant N\\(m,N)=1}} f(m)\sum_{\substack{[q_1^{\alpha_1},q_2^{\alpha_2}]\leqslant N/m\\ N-p=m[q_1^{\alpha_1},q_2^{\alpha_2}]}} f([q_1^{\alpha_1},q_2^{\alpha_2}])\ln q_1^{\alpha_1}\ln q_2^{\alpha_2}.$$

The part of the sum over those α_1, α_2 for which at least one is greater than 1 is estimated from trivial considerations; to the remaining part we apply the known estimates of the sieve method (^{3}) (ν_i are "almost primes")

$$\sum_{N=\nu_1+m\nu_2} 1\ll\frac{N}{m\ln^2 N/m}\prod_{\substack{p|N\\(p,m)=1}}\frac{1-1/p}{1-2/p},$$

which completes the proof.

Theorem 2. Let the assumptions of Theorem 1 be fulfilled, and let the series

$$\sum\frac{1-f(q)}{q},$$

extended over all q satisfying the condition $f(q)<1$, converge.

Then

$$\sum_{p\leq N} f(N-p)\asymp\frac{N}{\ln N}\exp\left\{\sum_{\substack{q\leq N\\(q,N)=1}}\frac{f(q)-1}{q}\right\},$$

where $\asymp$ is the Hardy sign.

Indeed,

$$\sum_{p\leq N} f(N-p)\geq\sum_{p\in M}{}' f(N-p),$$

where $p \in M$ if $N-p$ is square-free and $N-p$ is not divisible by any prime number q for which $f(q) < 1$. But on the set M the function $f(N-p)$ can be written as a sum over divisors of a positive function. Now the lower estimate becomes an immediate consequence of the "averaged" law of distribution of prime numbers in arithmetic progressions (^{1}).

Let us consider applications to the generalized Hardy–Littlewood problem. An asymptotic formula in this problem was derived by B. M. Bredikhin and Yu. V. Linnik (^{4}) by the ergodic method. As became clear after the work of A. I. Vinogradov (^{2}), the generalized Hardy–Littlewood problem reduces to estimating sums of values of Hecke characters (A. I. Vinogradov' s method made it possible to estimate the remainder term considerably more accurately). We shall show that sums of this type are easily estimated with the help of Theorem 1.

Theorem 3. Let $\chi(\omega)$ be a Hecke character of an imaginary quadratic field, of the form $\chi(\omega) = \chi_1(\omega)\chi_2^l(\omega)$, where $\chi_1(\omega)$ is a character of order k, and $\chi_2(\omega)$ is a fixed character of infinite order, l is an integer.

Then for all l, $0 < |l| \leq (\ln m)^c$, the inequality

$$\sum_{m=p+N(\omega)} \chi(\omega) < \frac{m(\ln\ln m)^c}{(\ln m)^{2-2/\pi}},$$

holds, and for $l = 0$ the inequality

$$\sum_{m=p+N(\omega)} \chi(\omega) \ll \frac{m(\ln\ln m)^c}{(\ln m)^{2-\xi/(k)}},$$

holds, where $\xi(k)$ is equal to

$$\frac{2}{k\tan\pi/k}, \qquad \frac{1}{k\sin\pi/2k}, \qquad \frac{2}{k\sin\pi/k}$$

respectively for $k \equiv 0$, $k \equiv 1, 3$, $k \equiv 2 \pmod 4$.

For the proof, introduce the multiplicative function

$$f(m) = \sum_{N(\omega)=m} \chi(\omega).$$

It plainly satisfies the conditions of Theorem 1, since

$$f(q^\alpha) \leq \tau(q^\alpha) = O(\alpha^c),$$

where $\tau(n)$ is the number of divisors of n. Further,

$$f(q) = |\chi(q) + \chi(\bar{q})| = |1 + \chi^2(\mathfrak{q})|,$$

since from $N(\omega) = q$ it follows that ω is a prime ideal of the first degree. Thus,

$$\sum_{m=p+N(\omega)} \chi(\omega) \ll \frac{m}{\ln^2 m} \ln\ln m \exp\left\{ \sum_{N(\mathfrak{q})\leq m} \frac{|1+\chi^2(\mathfrak{q})|}{2N(\mathfrak{q})} \right\}$$

($\mathfrak{q}$ is a prime ideal).

Further,

$$|1 + \chi^2(\mathfrak{q})| = \sum_{|d|\leq \ln m} b_d \chi^{2d}(\mathfrak{q}) + O\left(\frac{1}{\ln m}\right),$$

where the b_d are determined by the equality

$$|1 + e^{2\pi i\beta}| = \sum_{d=-\infty}^{+\infty} b_d e^{2\pi i\beta d}.$$

But since $2dl \ll (\ln m)^{c+1}$, using estimates for sums of Hecke characters (see (5)), we obtain

$$\sum_{N(\mathfrak{q})\leq m} \frac{\chi^{2d}(\mathfrak{q})}{N(\mathfrak{q})} = \begin{cases} \varepsilon_k(2d)\ln\ln m + O(1), & \text{for } l = 0, \\ \delta(d)\ln\ln m + O(\ln\ln\ln m), & \text{for } 0 < |l| \ll (\ln m)^c, \end{cases}$$

where $\varepsilon_k(x)$ ($\delta(x)$) is equal to 1 or 0 according as $x \equiv O(k)$ and $x \not\equiv O(k)$ (as $x = 0$ and $x \neq 0$); whence

$$\sum_{N(\mathfrak{q})\leq m} \frac{|1+\chi^2(\mathfrak{q})|}{N(\mathfrak{q})} = \begin{cases} b_0 \ln\ln m + O(\ln\ln\ln m), & \text{for } l \neq 0, \\ \left(\sum\limits_{2d\equiv O(K)} b_d\right) \ln\ln m + O(1), & \text{for } l = 0. \end{cases}$$

Summing the series, we complete the proof of the theorem.

Corollary. Let

$$Q(n) = \sum_{n=p+x^2+y^2} 1.$$

Then for $0 < \varphi_1 < \varphi_2 \leq 2\pi$ the following holds:

$$\sum_{\substack{n=p+x^2+y^2\\ \varphi_1<\operatorname{arc\,tg} x/y<\varphi_2}} 1 = \frac{\varphi_2-\varphi_1}{2\pi}\, Q(n) \left\{1 + O\left(\frac{(\ln\ln n)^c}{(\varphi_2-\varphi_1)(\ln n)^{1-2/\pi}}\right)\right\}.$$

V. I. Romanovskii Institute of Mathematics
Academy of Sciences of the Uzbek SSR

Tashkent State University
named after V. I. Lenin

Received
20 XI 1967

REFERENCES

1. M. B. Barban, *UMN*, 21, no. 1 (127), 51 (1966).
2. A. I. Vinogradov, *Matem. zametki*, 1, no. 2, 189 (1967).
3. K. Prachar, *Primzahlverteilung*, 1957.
4. B. M. Bredikhin, Yu. V. Linnik, *DAN*, 166, no. 6, 1267 (1966).
5. I. P. Kubilyus, *Matem. sborn.*, 31 (73), 507 (1952).

Note: Figure translations are in progress. See original paper for figures.