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ON χ-POINTS OF TOPOLOGICAL SPACES

MATHEMATICS

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Abstract

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MATHEMATICS

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ON χ-POINTS OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 30 III 1967)

As is known (see, for example, (5)), a point x of a topological space X is called a χ-point if there exists a sequence $\{x_i\}$ of points of the space X converging to it, i.e. the set $\{x_i\}$ is discrete in itself and the set $\{x_i\} \cup \{x\}$ is compact. The presence or absence of χ-points in a space X is a topological invariant of the space, but is almost unrelated to its structure. On the other hand, if some property is inherent in all continuous images of a space, then this necessarily imposes restrictions on its structure. In this connection, in the present note two classes of spaces are considered; membership in them is a topological invariant connected with χ-points.

All topological spaces are henceforth assumed to be completely regular.

Define the class K_1 as the maximal class of topological spaces, closed with respect to continuous mappings, each of which is either discrete or contains a χ-point. The class K_2 is the maximal class of topological spaces, closed with respect to continuous mappings, in which every nonisolated point is a χ-point.

1. By definition, both of the indicated classes are closed with respect to continuous mappings. This means, for example, that a space $X \in K_1$ if and only if it contains a χ-point or is discrete, and every continuous image of the space X has this property. The following theorems show the properties of the class K_1. Denote by $\mathfrak{K}_1$ the subclass of bicompact spaces of the class K_1.

Theorem 1. If $X \in \mathfrak{K}_1$, $f : Y \to X$ is an open continuous mapping of a bicompactum Y onto a bicompactum X, and $f^{-1}(x) \in \mathfrak{K}_1$ for every $x \in X$, then $Y \in \mathfrak{K}_1$.

The proof of this theorem is based on the following lemma:

Lemma 1. If X, Y, Z are bicompacta, $f : Y \to X$ is an open mapping onto X, $\varphi : Y \to Z$ is a continuous mapping, and the set $\varphi f^{-1}(x)$ is finite for every $x \in X$, then the decomposition $\mathfrak{M}$: $x_1 \sim x_2 \iff \varphi f^{-1}(x_1) = \varphi f^{-1}(x_1)$ of the space X is continuous.

From Theorem 1 there follows immediately

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Theorem 2. If $X_i \in \mathfrak{K}_1$, $i = 1, 2, \dots, n$, then

$$\prod_{i=1}^{n} X_i \in \mathfrak{K}_1.$$

With the help of this theorem and some auxiliary lemmas on mappings, one of the main theorems is proved.

Theorem 3. If $X_\gamma \in \mathfrak{K}_1$ for every $\gamma \in \Gamma$, where Γ is some set of indices, then

$$\prod_{\gamma \in \Gamma} X_\gamma \in \mathfrak{K}_1.$$

For what follows, the following is important.

Theorem 4. If $X \in K_1$, then $\beta X \in K_1$, where βX denotes the Stone–Čech compactification of the space X.

Let X be a topological space. The quasicomponent of a point qx, as is known, is called the intersection of all open-and-closed sets containing the point x, and the quasicomponent qx coincides with the trace in the space X of the component of the point x in the space βX. Consider the set ηX of quasicomponents as a topological space with the topology inherited in the space of components of points of the space βX. As is easy to see, the mapping $\varphi(x) = qx$, which assigns to each point x of the space X its quasicomponent qx in the space of quasicomponents, is continuous and is a mapping onto.

Theorem 5. *If the topological space $X \in K_1$, then the space ηX of its quasicomponents is pseudocompact.*

A space X is called pseudocompact if every continuous real-valued function on it is bounded.

With the aid of Theorems 3, 4, and 5 one proves Theorem 6, which completely describes the multiplicative properties of the class K_1.

Theorem 6. *Let Γ be some set of indices. The product $\prod_{\gamma \in \Gamma} X_\gamma$ belongs to the class K_1 if and only if $X_\gamma \in K_1$ for all $\gamma \in \Gamma$ and the product of the spaces of quasicomponents of the factors $\prod_{\gamma \in \Gamma} \eta X_\gamma$ is pseudocompact.*

Note that

$$\prod_{\gamma \in \Gamma} \eta X_\gamma = \eta \prod_{\gamma \in \Gamma} X_\gamma.$$

2. We now turn to the consideration of spaces of the class K_2. Analogously to the preceding item, define the class $\mathfrak{K}_2$ as the subclass of bicompact spaces of the class K_2.

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For the class $\mathfrak{K}_2$ one first proves finite multiplicativity, and then

Theorem 7. *The space* $\prod_{\gamma\in\Gamma} X_\gamma \in \mathfrak{K}_2$ *if and only if* $X_\gamma \in \mathfrak{K}_2$ *for all* $\gamma \in \Gamma$, *where* Γ *is an arbitrary set of indices.*

Let X be a topological space. Denote by X' the space obtained by adding to the space X an isolated point, and denote by P_2 the subclass of those spaces X for which $X' \in K_2$.

Theorem 8. *If* $X \in P_2$, *then* X *is a pseudocompact space.*

Theorem 9. *If* X *is a disconnected space and* $X \in K_2$, *then* $X \in P_2$.

It is also obvious that $\mathfrak{K}_2 \subset P_2$.

Denote by $\mathfrak{U}_2$ the class of connected spaces belonging to K_2.

Theorem 10. *If* $X_\gamma \in \mathfrak{U}_2$ *for all* $\gamma \in \Gamma$, *then* $\prod_{\gamma\in\Gamma} X_\gamma \in \mathfrak{U}_2$.

With the aid of Theorems 7, 8, and 10 one proves Theorem 11, which completely solves the question of multiplicativity of the class.

Theorem 11. *The space* $\prod_{\gamma\in\Gamma} X_\gamma \in K_2$ *if and only if at least one of the following two conditions is satisfied:*

1) $X_\gamma \in U_2$ for all $\gamma \in \Gamma$.
2) $X_\gamma \in P_2$ for all $\gamma \in \Gamma$ and $\prod_{\gamma\in\Gamma} X_\gamma$ is pseudocompact.

Note that the classes K_2, P_2, $\mathfrak{K}_2$, $\mathfrak{U}_2$ are closed with respect to continuous mappings, and the classes $\mathfrak{K}_2$ and $\mathfrak{U}_2$ also with respect to products in any transfinite number. The question of multiplicativity of the classes P_2 and K_1 remains open.

3. We now pass to the consideration of the spaces $\exp X$ of closed subsets of a bicompactum X in the Vietoris topology. As is known, if X is a bicompactum, then the space $\exp X$ is also a bicompactum.

Let σ be some maximal chain of closed subsets of the bicompactum X under inclusion.

Theorem 12. *The topology of the set* $\sigma \subset \exp X$, *induced by the space* $\exp X$, *coincides with the order topology in* σ, *whose order relation is the inclusion relation.*

Theorem 13. A maximal chain σ is a closed subset of the space $\exp X$.

Theorem 14. If σ is an ordered bicompactum, then $\sigma \in K_1$.

From the formulated theorems it follows that in the space $\exp X$ there must be sufficiently many χ-points. Indeed:

Theorem 15. If $H \subset X$, H is an infinite set, $[H] = H$, then $\hat{H} \in \exp X$ is a χ-point, where the symbol $\hat{H}$ denotes the point of the space $\exp X$ corresponding to the set H.

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With the aid of this theorem one proves

Theorem 16. For every bicompactum X, the bicompactum $\exp X \in \mathfrak{K}_1$.

Theorem 17. The bicompactum $\exp X \in \mathfrak{K}_2$ if and only if $X \in \mathfrak{K}_2$.

In the proof of this theorem the following is also used:

Lemma 2. If $\varphi : X \to Y$ is a continuous mapping of a bicompactum X onto a bicompactum Y, then there exists a continuous mapping $\Phi : \exp X \to \exp Y$ that coincides with the mapping φ on the set of one-point subsets of the space X.

Theorem 18. A point $x \in X$ is a χ-point if and only if the one-point subset $\{x\}$ is a χ-point in the space $\exp X$.

4. Let us now consider the relations between the known classes of spaces and the classes K_1 and K_2.

For Fréchet-Urysohn spaces, and therefore for metrizable spaces and spaces with the first axiom of countability, the question of belonging to the class K_1 is completely resolved.

Theorem 19. If X is a Fréchet-Urysohn space, then $X \in K_1$ if and only if the space ηX of its quasicomponents is pseudocompact.

For the condition of pseudocompactness of the space of quasicomponents we have

Theorem 20. The space of quasicomponents ηX of the space X is pseudocompact if and only if there exists no continuous mapping $\varphi : X \to N$ of the space X onto a countable discrete space N.

For the class K_2 we have

Theorem 21. If X is a Fréchet-Urysohn bicompactum, then $X \in \mathfrak{K}_2$.

Theorem 22. If X is a Fréchet-Urysohn space, then $X \in K_2$ if and only if the space ηX of quasicomponents of the space X belongs to the class K_2.

For ordered spaces, membership in the classes K_1 and K_2 is expressed by the following theorems.

Theorem 23. An ordered space $X \in K_1$ if and only if the space ηX of its quasicomponents is pseudocompact.

Theorem 24. An ordered space $X \in K_2$ if and only if X is a space with the first axiom of countability and the space ηX of its quasicomponents is bicompact.

We note that for ordered spaces the components of points coincide with their quasicomponents.

Since finite discrete spaces belong to the class $\mathfrak{K}_2$, it follows from Theorems 3 and 7 that the Efimov-Katětov theorem holds:

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Corollary 1. Every non-isolated point x of a dyadic bicompactum X is a χ-point.

Corollary 2. All dyadic bicompacta belong to the class $\mathfrak{K}_2$.

Examples. The paper gives examples showing that there exist connected bicompacta of any cardinalities, weights, and dimensions that have no χ-points; that the property of belonging to the classes K_1 and K_2 is not inherited either by closed sets of type G_δ or by canonical closed sets.

It is possible to apply the results set out above to the investigation of the possibility of mappings of some topological spaces onto others.

In particular, the following corollary of Theorem 16 holds:

Corollary 3. *If X is a bicompactum not belonging to the class $\mathfrak{R}_1$, then there is no continuous mapping of the space of closed subsets of this bicompactum onto the bicompactum itself.*

In particular, this holds for extremely disconnected bicompacta and their closed subsets.

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