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TO THE
APPROXIMATE
SOLUTION OF A
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PROBLEM**

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Abstract

Full Text

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MATHEMATICS

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APPLICATION OF A CONTINUOUS ANALOGUE OF NEWTON' S METHOD TO THE APPROXIMATE SOLUTION OF A NONLINEAR BOUNDARY-VALUE PROBLEM

(Presented by Academician N. N. Bogolyubov, 29 VI 1967)

In paper ⁽¹⁾ continuous analogues of certain iterative processes for solving nonlinear functional equations in a Banach space were considered. An existence theorem for a solution was given, proved with the aid of a continuous analogue of Newton' s method.

Let us note that the continuous analogue of Newton' s method can be effectively applied to the numerical solution of certain problems when the existence of a solution is known or can be established in another way. In papers ^(2, 3) conditions for the applicability of this method in the approximate solution of a boundary-value problem for one nonlinear second-order differential equation were formulated, under the assumption that a solution of the problem exists. The possibility of approximately solving nonlinear boundary-value problems by means of the indicated method was also pointed out in ⁽⁴⁾.

In the present note we give conditions for the possibility of solving, by means of the indicated method, a nonlinear functional equation in a Banach space, if a solution of the equation exists. An approximate method for finding this solution is considered. On the basis of these assertions, an approximate method is justified for finding solutions of the ordinary differential equation

$$f(x, y, y', y'') = 0, \tag{1}$$

satisfying the boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0, \\ \alpha_i^2 + \beta_i^2 > 0, \quad i &= 1, 2. \end{aligned} \tag{2}$$

One of the possible numerical algorithms for solving problem (1)–(2) is presented.

1. Let $\varphi(x)$ be a continuous function mapping a Banach space X into a Banach space Y . Consider the equation

$$\varphi(x) = 0. \quad (3)$$

Theorem 1. *Let equation (3) have a unique solution x^* in an open domain D of the space X . Suppose that in D there exist continuous Fréchet derivatives $\varphi'(x)$, $\varphi''(x)$, as well as the inverse operator $\varphi'(x)^{-1}$, for which the inequality*

$$\|\varphi'(x)^{-1}\| \leq B \quad (4)$$

holds.

Then there exists a sphere $S : \|x - x^*\| \leq \varepsilon$, belonging to the domain D , such that for any $x_0 \in S$ the differential equation

$$x'_t = -\varphi'(x)^{-1}\varphi(x), \quad (5)$$

where t is a real parameter, with the initial condition

$$x(0) = x_0 \quad (6)$$

has a unique solution $x(t)$ for $0 \leq t < \infty$, and $\lim_{t \rightarrow \infty} x(t) = x^*$.

Proof. In view of the relation $\lim_{x \rightarrow x^*} \|\varphi(x)\| = 0$, one can choose $\varepsilon > 0$ such that, for any $x_0 \in S : \|x - x^*\| \leq \varepsilon$, the following will hold:

$$\|\varphi(x_0)\| \leq \delta,$$

where $\delta > 0$ is sufficiently small and $S \subset D$. Choose δ so small that the sphere $R : \|x - x_0\| \leq B\delta$ will belong to D for any $x_0 \in S$. The conditions of the theorem ensure the existence of a unique solution of problem (5)–(6) on a sufficiently small interval of variation of the parameter t (⁵). To complete the proof it suffices to carry out arguments similar to those given in Theorem 1 of paper (¹).

2. Let X be a Banach space, and let $\psi(x)$ be a function mapping X into itself. Suppose that, in some open domain $D \subset X$, $\psi(x)$ satisfies the Lipschitz condition and $\|\psi(x)\| \leq M$.

Consider the differential equation

$$x'_t = \psi(x), \quad x(0) = x_0 \in D, \quad (7)$$

where t is a real parameter. On some interval of variation of t , $0 \leq t \leq \sigma$, there exists ⁽⁵⁾ a unique solution of equation (7) belonging to the domain D . Divide $[0, \sigma]$ into n parts by the nodal points $\sigma_0, \sigma_1, \dots, \sigma_n$, where $\sigma_i = \sigma_{i-1} + \tau_i$, $i = 1, 2, \dots, n$, $\sigma_0 = 0$, $\sigma_n = \sigma$. Let $\tau = \max \tau_i$. We shall call the sequence of elements \bar{x}_i , $i = 1, 2, \dots, n$, obtained by means of the relations

$$\bar{x}_0 = x_0, \quad \bar{x}_i = \bar{x}_{i-1} + \psi(\bar{x}_{i-1})\tau_i, \quad i = 1, \dots, n, \quad (8)$$

an approximate solution of equation (7) obtained by Euler' s method.

Theorem 2. Let $\psi(x)$ satisfy the Lipschitz condition in D ; let $\|\psi(x)\| \leq M$, and let the solution of equation (7) for $0 \leq t \leq \sigma$ lie in D . Suppose, further, that

$$\tau \leq K\sigma/n, \quad (9)$$

where $K \geq 1$ does not depend on n .

Then, as $\tau \rightarrow 0$, the solution obtained by Euler' s method converges to $x(t)$ on $[0, \sigma]$.

Proof. It is easy to show that, at the nodal point σ_i , the residual of the solution $\Delta_i = \|x_i - \bar{x}_i\|$ satisfies the relation

$$\Delta_i \leq \Delta_{i-1}(1 + L\tau) + \frac{1}{2}LM\tau^2,$$

where L is the Lipschitz constant. Applying the method set forth in ⁽⁶⁾, we obtain

$$\Delta_i \leq \frac{1}{2}M\tau(e^{LK\sigma} - 1), \quad i = 1, 2, \dots, n.$$

From the last estimate, as $\tau \rightarrow 0$, we obtain convergence of the approximate solution to the exact one, which completes the proof.

- Let Y be the set of twice continuously differentiable functions $y(x)$ on $[a, b]$ satisfying the boundary conditions (2), whose second derivatives satisfy the Lipschitz condition on $[a, b]$. We introduce the norm for $y(x)$ as follows:

$$\|y(x)\| = \sum_{k=0}^2 \max_{a \leq x \leq b} |y^{(k)}(x)| + \inf L_{y''}.$$

Let, further, Z be the set of functions satisfying a Lipschitz condition on $[a, b]$, for which

$$\|z(x)\| = \max_{a \leq x \leq b} |z(x)| + \inf L_z.$$

It is easy to see that Y and Z are Banach spaces.

Theorem 3. *Let inside a closed domain $D_Y \subset Y$ there be contained the unique solution $y^*(x)$ of problem (1)–(2). Suppose that $f(x, y, y', y'')$ has in D_Y continuous partial derivatives up to the second order inclusive and*

$$|f'_{y''}| \geq a.$$

Let, further, the boundary-value problem

$$f'_{y''}(x, y, y', y'')v'' + f'_{y'}(x, y, y', y'')v' + f'_y(x, y, y', y'')v = 0, \quad (10)$$

$$\alpha_1 v(a) + \beta_1 v'(a) = 0,$$

$$\alpha_2 v(b) + \beta_2 v'(b) = 0 \quad (11)$$

have only the trivial solution for any function $y(x) \in D_Y$. Then there exists $\varepsilon > 0$ such that for any function $y_0(x) \in D_Y$ satisfying the condition

$$\|f(x, y_0, y'_0, y''_0)\|_z \leq \varepsilon,$$

the system with respect to the functions $y(x, t), v(x, t)$

$$\begin{aligned} f'_{y''}(x, y, y'_x, y''_{xx})v''_{xx} + f'_{y'}(x, y, y'_x, y''_{xx})v'_x + f'_y(x, y, y'_x, y''_{xx})v = \\ = -f(x, y, y'_x, y''_{xx}), \quad y'_t = v \end{aligned} \quad (12)$$

with the conditions

$$\alpha_1 v(a, t) + \beta_1 v'_x(a, t) = 0,$$

$$\alpha_2 v(b, t) + \beta_2 v'_x(b, t) = 0, \quad (13)$$

$$y(x, 0) = y_0(x)$$

has a unique solution and

$$\lim_{t \rightarrow \infty} \|y(x, t) - y^*(x)\|_y = 0.$$

For the proof of the theorem it is sufficient to verify that for the operator $f(y) = f(x, y, y', y'')$, mapping Y into Z , the conditions of Theorem 1 are satisfied.

4. Consider the following scheme for the numerical solution of problem (1)–(2). Divide the half-strip $a \leq x \leq b$, $0 \leq t < \infty$ by straight lines parallel to the x -axis, $t = t_i$, separated from one another by τ_i . We replace the second equation of system (12) by the approximate difference relation

$$y(x, t_{i+1}) = y(x, t_i) + \tau_{i+1}v(x, t_i). \quad (14)$$

On the layer $t = t_i$ we solve the linear ordinary differential equation with respect to $v(x, t_i)$

$$f'_{y''_i}v''(x, t_i) + f'_{y'_i}v'(x, t_i) + f'_{y_i}v(x, t_i) = -f_i, \quad (15)$$

where the known function $y(x, t_i)$ is substituted into the coefficients, with the boundary conditions

$$\alpha_1v(a, t_i) + \beta_1v'(a, t_i) = 0,$$

$$\alpha_2v(b, t_i) + \beta_2v'(b, t_i) = 0. \quad (16)$$

For $i = 0$ the initial function $y(x, 0)$ must satisfy conditions (2) and be sufficiently close to the sought solution. Relations (14)–(16) make it possible successively to determine the functions $v(x, t_i)$, $y(x, t_{i+1})$. We note that this scheme is a realization of Euler's method-

to solving the differential equation

$$y'_t = -f'(y)^{-1}f(y), \quad y(0) = y_0, \quad f(y) = f(x, y, y', y'').$$

The convergence of the method is proved by Theorem 2.

In the course of the computations, the step in the parameter t can be chosen in an optimal manner. The boundary-value problem (15)–(16) can be solved by any known numerical method. From the point of view of implementation on a

computer, the sweep method is convenient for the difference equation obtained by approximating the differential operator by a finite-difference one on a uniform grid of nodes of the interval $[a, b]$.

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