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Abstract

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ON THE LYAPUNOV-GÜNTER THEOREMS FOR SPECIAL HEAT POTENTIALS

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In paper ([1]) (see also ([2], [3])) a theory was constructed for the smoothness of heat potentials of a simple and a double layer with densities distributed on noncylindrical surfaces, entirely analogous to the classical Lyapunov-Günter theory (see ([4])) for harmonic potentials. In ([5]) (see also ([6])) the construction of a similar theory was begun for the special heat potentials P and Q (introduced by M. Pani in ([7])), which play an important role in the theory of boundary-value problems with an oblique derivative for a parabolic equation of the second order. The present note, consisting of two sections, contains a complete and systematic investigation in Hölder smoothness spaces of the special heat potential of a simple layer $P[\varphi]$ and the heat potential $Q[\varphi]$, which is the derivative of the potential $P[\varphi]$ in an oblique direction. In § 1 the improving properties of the direct values $\overline{Q}[\varphi]$ on noncylindrical surfaces of various smoothness are studied, while in § 2 the smoothness of the heat potential $P[\varphi]$ in the closed domain $\overline{D}_T$, having Γ as its lateral boundary, is studied. The notation and definitions of the author' s papers ([1], [5]) are used in the note.

Let D_T be a bounded domain of the $(n+1)$-dimensional Euclidean space $(x,t) \equiv (x_1, x_2, \dots, x_n; t)$, situated between two hyperplanes $t=0$ and $t=T>0$ and having an n-dimensional noncylindrical surface Γ as its lateral boundary. Consider the heat potentials

$$P(\bar{x},t) \equiv P[\varphi] \equiv \int_0^t d\tau \iint_{\Gamma_\tau} P(\bar{x},t;y,\tau)\varphi(y,\tau)\, d\sigma_y(\tau), \qquad (\bar{x},t) \in D_T,$$

$$Q(\bar{x},t) \equiv Q[\varphi] \equiv \partial P(\bar{x},t)/\partial\nu(x,t), \qquad (x,t) \in \Gamma_t;$$

Γ_τ is the section of the surface Γ by the hyperplane $t=\tau$, on which a field of directions is given with unit vector
$\nu(x,t) = \{\nu_1(x,t), \dots, \nu_n(x,t;0)\}$, lying in the section $\Omega_t \equiv D_T \cap \{t=t\}$ and making an acute angle not exceeding $\pi/2 - d_0$ ($d_0>0$) with the normal $N(x,t)$

interior with respect to Ω_t at the point $(x,t) \in \Gamma_t$. $P(\bar{x},t;y,\tau)$ is the special fundamental solution, introduced by M. Pani in ([7]), of the n-dimensional heat-conduction equation corresponding to the field of oblique directions $\nu(x,t)$.

§ 1. **Improving properties of the direct values of the heat potential $\overline{Q}[\varphi]$.** The notation α', α^0, α^*, β' from ([5]) will be used.

Theorem 1. Suppose that for Γ, ν, and φ the following conditions are satisfied: Γ is of type

$$\Pi^{m+1,\,\alpha,\,\alpha/2}_{2m+1,\,1,\,(1+\alpha)/2}, \qquad 0<\alpha\le 1,$$

$$\nu_j \in H^{m,\,1,\,(1+\beta)/2}_{2m+1,\,\beta,\,\beta/2}(\Gamma), \qquad 0<\alpha\le\beta\le 1,$$

$$\cos(\nu(x,t),N(x,t)) \ge d > 0, \qquad (x,t)\in\Gamma_t, \quad d=\text{const}, \quad 0\le t\le T, \tag{1}$$

$$\varphi \in H^{m,\,\alpha,\,\alpha/2}_{2m-1,\,1,\,(1+\alpha)/2}(\Gamma),$$

where

$$\left|\partial^k\varphi(y,\tau)/\partial\tau^k\right| \le \left|\partial^m\varphi/\partial t^m\right|_\alpha \tau^{m-k+\alpha/2},$$
$$(y,\tau)\in\Gamma, \qquad k=0,1,2,\dots,m.$$

Then for $m=1,2,\dots$ (for $m=0$ see Lemma 4 (5)) for $\overline{Q}(x,t)$ one has

$$\overline{Q} \in H^{m,1,(1+\alpha^*)/2}_{2m+1,\alpha,\alpha^*/2}(\Gamma),$$

where the Hölder constants have the form $(C)|\varphi|_{2m+\alpha}$,

$$\left|\partial^k\overline{Q}(x,t)/\partial t^k\right| \le (C)|\varphi|_{2m+\alpha} t^{m-k+(1+\alpha)/2}, \qquad (x,t)\in\Gamma, \qquad k=0,1,\dots,m,$$

and when

$$p=0,1,2,\dots,m, \qquad l_j=0,1,\dots,2(m-p), \qquad j=1,2,\dots,k,$$
$$\sum_{j=1}^{k} l_j = 2(m-p), \tag{2}$$

where $k=n-1$,

$$\left|\partial^{2m-p+l}\overline{Q}(x,t)/\partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}} \partial x_i^l\right| \leq (C)|\varphi|_{2m+\alpha} t^{(1+\alpha-l)/2}, \qquad l = 0, 1.$$

Theorem 2. Suppose that for Γ, ν, and φ the following conditions are satisfied: Γ is of type

$$\mathcal{L}_{2m+3,\alpha,\alpha/2}^{m+1,1,(1+\alpha)/2}, \qquad 0 < \alpha \leq 1 \quad (m = 0, 1, 2, \ldots),$$

$$\nu_j \in H_{2m+1,1,(1+\beta)/2}^{m+1,\beta,\beta/2}(\Gamma), \qquad 0 < \alpha \leq \beta \leq 1, \qquad j = 1, 2, \ldots, m$$

(with (1) satisfied),

$$\varphi \in H_{2m+1,\alpha,\alpha/2}^{m,1,(1+\alpha)/2}(\Gamma),$$

$$\left|\partial^{k+l}\varphi(y,\tau)/\partial \tau^k \partial y_i^l\right| \leq \left|\partial^m \varphi/\partial t^m\right|_{1+\alpha} \tau^{m-k+(1+\alpha-l)/2}$$

$$(y,\tau) \in \Gamma, \quad k = 0, 1, 2, \ldots, m; \quad i = 1, 2, \ldots, n-1; \quad l = 0 \text{ for } k < m,$$
$$l = 0, 1 \text{ for } k = m.$$

Then for $m = 0, 1, 2, \ldots$ for $\overline{Q}(x,t)$ one has

$$\overline{Q} \in H_{2m+1,1,(1+\alpha^*)/2}^{m+1,\alpha^*,\alpha^*/2}(\Gamma),$$

where the Hölder constants have the form $(C)|\varphi|_{2m+1+\alpha}$,

$$\left|\partial^k \overline{Q}(x,t)/\partial t^k\right| \leq (C)|\varphi|_{2m+1+\alpha} t^{m-k+\alpha/2}, \qquad k = 0, 1, \ldots, m,$$

and, when (2) is satisfied, where $k = n-1$,

$$\left|\partial^{2m+1-p}\overline{Q}(x,t)/\partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}} \partial x_i\right| \leq (C)|\varphi|_{2m+1+\alpha} t^{(1+\alpha)/2},$$

$$\left(\left|\partial^{2m+1-p}\overline{Q}(x,t)/\partial t^{p+1} \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}\right|,\right.$$

$$\left.\left|\partial^{2m+2-p}\overline{Q}(x,t)/\partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}} \partial x_i \partial x_j\right|\right) \leq (C)|\varphi|_{2m+1+\alpha} t^{\alpha/2}.$$

§ 2. Smoothness of the special heat potential of a simple layer $P[\varphi]$ in the closed domain $\overline{D}_T$

Theorem 3. Suppose that for Γ, ν the conditions of Theorem 1 are satisfied,

$$\varphi \in H^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}(\Gamma) \qquad \text{for } m = 0, 1, 2, \dots,$$

$$\partial^k \varphi(y,0)/\partial\tau^k \equiv 0, \qquad k = 0, 1, \dots, m, \qquad (y,0) \in \Gamma_0,$$

and, when (2) is satisfied, where $k = n-1$,

$$\left(\left|\partial^{2m-p+l}\varphi(y,\tau)/\partial\tau^p \partial y_1^{l_1} \cdots \partial y_{n-1}^{l_{n-1}} \partial y_i^l\right| \le (C)|\varphi|_{2m+1+\alpha}\tau^{(1+\alpha-l)/2}, \qquad l = 0, 1.$$

Then

$$P \in H^{m+1,\,\alpha',\,\alpha'/2}_{2m+1,\,1,\,(1+\alpha)/2}(\overline{D}_T),$$

where the Hölder constants have the form $(C)|\varphi|_{2m+1+\alpha}$, and (see (2), where $k = n$)

$$\left|\partial^{2m+1+l-p}P(\bar{x},t)/\partial t^p \partial\bar{x}_1^{l_1} \cdots \partial\bar{x}_n^{l_n} \partial\bar{x}_i\right| \le (C)|\varphi|_{2m+1+\alpha}t^{(1+\alpha-l)/2}, \quad l = 0, 1,$$

$$\left|\partial^{2m+1-p}P(\bar{x},t)/\partial t^{p+1} \partial\bar{x}_1^{l_1} \cdots \partial\bar{x}_n^{l_n}\right| \le (C)|\varphi|_{2m+1+\alpha}t^{\alpha/2}, \quad (\bar{x},t) \in \overline{D}_T$$

(for $m = 0$, Theorem 3 coincides with Lemma 6 of ([5])).

Theorem 4. Suppose that for Γ and ν the conditions of Theorem 2 are fulfilled, and

$$\varphi \in H^{m+1,\,\alpha,\,\alpha/2}_{2m+1,\,1,\,(1+\alpha)/2}(\Gamma) \quad \text{for } m = 0, 1, 2, \dots,$$

where

$$\partial^k \varphi(y,0)/\partial\tau^k = 0, \quad k = 0, 1, 2, \dots, m+1,$$

and (see (2), where $k = n-1$)

$$\left|\partial^{2m+1+l-p}\varphi(y,\tau)/\partial\tau^p \partial y_1^{l_1} \cdots \partial y_{n-1}^{l_{n-1}} \partial y_i \partial y_j\right| \le (C)|\varphi|_{2m+2+\alpha}\tau^{(1+\alpha-l)/2}, \quad l = 0, 1;$$

$$\left|\partial^{2m+1-p}\varphi(y,\tau)/\partial\tau^{p+1}\partial y_1^{l_1}\cdots\partial y_{n-1}^{l_{n-1}}\right| \le (C)|\varphi|_{2m+2+\alpha}\tau^{\alpha/2}.$$

Then

$$P \in H_{2m+3,\,\alpha',\,\alpha'/2}^{m+1,\,1,\,(1+\alpha)/2}(\overline{D}_T),$$

where the Hölder constants have the form $(C)|\varphi|_{2m+2+\alpha}$, and, when (2) is fulfilled, where $k = n$ and m is replaced by $m+1$,

$$\left|\partial^{2(m+1)+l-p}P(\bar{x},t)/\partial t^p\partial\bar{x}_1^{l_1}\cdots\partial\bar{x}_n^{l_n}\partial\bar{x}_i\right| \le (C)|\varphi|_{2m+2+\alpha}t^{(1+\alpha-l)/2}, \quad l = 0, 1.$$

The proofs of Theorems 1-4 are carried out by the methods of the papers ([2], [3], [6]).

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