CONSTRUCTION OF A SEQUENCE OF STRONGLY INDEPENDENT SUPERINTUITIONISTIC PROPOSITIONAL CALCULI

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Abstract

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CONSTRUCTION OF A SEQUENCE OF STRONGLY INDEPENDENT SUPERINTUITIONISTIC PROPOSITIONAL CALCULI

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By a **superintuitionistic logic** we shall mean a set of propositional formulas in the language with four connectives $\&, \supset, \vee, \neg$, containing the intuitionistic axioms and closed under the rules of substitution and *modus ponens*. A logic is a **calculus** if it is obtained by adding to the intuitionistic axioms a finite number of additional ones and closing under the rules of inference.

Logics in a sequence $N_1, \ldots, N_k, \ldots$ will be called **strongly independent** (see ([1])) if, for every i, N_i is not contained in the logic generated by all N_j with $j \neq i$. The models of the intuitionistic propositional calculus studied in ([2]) under the name of pseudo-Boolean algebras will be called Brouwer algebras* . A Gödelian Brouwer algebra (see ([3])) will be called (in the finite case) a Brouwer algebra with a unique immediate predecessor of the greatest element** . By a Λ-algebra we shall hereafter mean a finite Gödelian Brouwer algebra α in which there is a system of three generators a_1, a_2, a_3 and the following relations hold:

$$\begin{aligned} 1) \quad & \neg(a_1 \& a_2) = \neg(a_2 \& a_3) = \neg(a_1 \& a_3) = (\neg\neg a_3 \supset a_3) = \\ & = (\neg\neg a_2 \supset a_2) = \neg\neg(a_1 \vee a_2 \vee a_3) = V; \\ 2) \quad & \neg a_3 \vee \neg a_2 \vee (\neg\neg a_1 \supset a_1) = \omega. \end{aligned}$$

Here V is the greatest element of α, and ω is its unique immediate predecessor.

In ([3]), for every Gödelian Brouwer algebra α a formula X_α was introduced with the property that an arbitrary formula F is refutable on α (see ([3]), bearing in mind that in α the greatest element V is singled out) if and only if $F \vdash X_\alpha$ (Theorem 2 of ([3]); the symbol $\vdash$ denotes derivability of X_α from F using the intuitionistic axioms, *modus ponens*, and substitution). In ([3]) the following ordering of Gödelian Brouwer algebras is also introduced (isomorphic algebras being identified): $\alpha \leqslant \beta$ if and only if $X_\alpha \vdash X_\beta$. The following two lemmas ensure the possibility of constructing a sequence of strongly independent calculi.

Lemma 1. *Two Λ-algebras are either incomparable or isomorphic.*

Diagram 1

Figure 1: Diagram 1

Lemma 2. *There exist* Λ*-algebras with an arbitrarily large number of elements.*

As for the second of these lemmas, the desired Λ-algebras can be constructed as shown in diagram 1 (the distinguished element V is at the top).

It follows from these lemmas that there exists a sequence $\{\alpha_i\}$ of incomparable Λ-algebras. It is convenient for us to assume that $\{\alpha_i\}$ even contains all noniso-

* In ([3]) they were called implicative structures, which was not quite exact, since it did not reflect the presence of the operation $\neg$. The terminology has not yet become fixed, but the term chosen in ([2]) seems unsuccessful.

** In ([2]) an equivalent notion of a strongly compact pseudo-Boolean algebra is introduced.

morphic Λ-algebras (being a Λ-algebra is a decidable property, as is isomorphism of finite Brouwer algebras). Then X_{α_i} and a_i determine the desired calculi N_i.

Indeed, for $j \neq i$ all X_α are generally valid on a_i, in view of the incomparability of a_j and a_i, while X_α is refuted on a_i, whence X_{α_i} cannot be contained in the logic generated by all X_{α_j} $(j \neq i)$.

Diagram 1

Corollary 1 (classical). *There exists a continuum of distinct superintuitionistic logics.*

Corollary 2. *There exists a superintuitionistic logic that is not a calculus. (Such is the logic L generated by all X_α.)*

Corollary 3. *There exists a strictly increasing sequence M_i of superintuitionistic calculi (M_i is determined by the axiom* $\mathop{\&}\limits_{i=1}^{j} X_{\alpha_i}$*).*

The logic L also has the following property. The formula

$$
\begin{gathered}
F \equiv \neg(a_1 \& a_2) \;\&\; \neg(a_2 \& a_3) \;\&\; \neg(a_1 \& a_3) \;\& \\
\&(\neg\neg a_3 \supset a_3) \;\&\; (\neg\neg a_2 \supset a_2) \;\&\; \neg\neg(a_1 \vee a_2 \vee a_3) \;\supset \\
\supset \neg a_3 \vee \neg a_2 \vee (\neg\neg a_1 \supset a_1)
\end{gathered}
$$

is not contained in L, but there is no finite Brouwer algebra a on which F would be refuted while all formulas of L would be generally valid.

The first assertion follows from the fact that in a derivation of F we can use only a finite number of formulas X_{α_i}, but, taking a_j distinct from all such a_i, we have: F is refuted on a_j, while all X_{α_i} are generally valid on a_j.

If, however, F is refuted on some finite Brouwer algebra α on which all Λ_{α_i} are generally valid, then, after applying a finite number of times the passage from the algebra to some homomorphic image of it or to some subalgebra of it, we could find a Λ-algebra β with the same properties, but among the a_i there is an algebra isomorphic to it, so that $X_{\alpha_i} \in L$ and X_{α_i} is refuted on β, which contradicts the assumption.

It follows from this theorem that Theorem 3, 4 formulated in ([1]) is false. The error in the proof (independently noticed by A. V. Kuznetsov and by me) consists in the implicit application of the relation

$$\overline{\left(L \cap \bigcup_{i=1}^{\infty} L_i\right)} = \bigcap_{i=1}^{\infty} \overline{(L \cup L_i)}.$$

(Here L, L_i are logics, and the bar denotes closure with respect to the rules of inference), which is not always satisfied, as follows from the present work.

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Note: Figure translations are in progress. See original paper for figures.