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Abstract

Full Text

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MATHEMATICS

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ON THE CONSTRUCTION OF EXTREMAL TRIGONOMETRIC POLYNOMIALS OF MIXED TYPE

(Presented by Academician S. L. Sobolev on 24 V 1967)

Put

$$T_{l,m}(\theta) = \sum_0^l a_k \cos k\theta + \sum_1^m b_k \sin k\theta = c_l(\theta) + s_m(\theta)$$

under the condition

$$\max |T_{l,m}(\theta)| = 1$$

(a normalized polynomial). In the special cases $T = c_l(\theta)$ or $T = s_m(\theta)$ it was proved ([1]) that all of them are obtained analytically from the corresponding algebraic polynomials. Our problem is to show that every $T_{l,m}(\theta)$ of mixed type can also be constructed with the aid of extremal algebraic polynomials.

Theorem 1. To every normalized $T_{l,m}(\theta)$ there corresponds a completely determined pair of normalized analytic functions

$$\Phi_{1,2}(y) = Q_l(y) \pm P_{m-1}(y)\sqrt{1-y^2},$$

each normalized on $[-1, +1]$; conversely, to every such pair $\Phi_{1,2}(y)$ there corresponds a unique $T_{l,m}(\theta)$.

1) Let $T_{l,m}(\theta)$ be given. Without loss of generality one may suppose that

$$|T_{l,m}(\pm\pi)| < 1 \quad \text{and} \quad |T_{l,m}(0)| < 1$$

(otherwise we take $T_{l,m}(\theta + \varphi_0)$). We have

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$$c_l(\theta) = \sum_0^l A_k \cos^k \theta; \qquad s_m(\theta) = \sum_0^{m-1} B_k \cos^k \theta \sin \theta,$$

where (A_k) and (B_k) are determined uniquely.

Splitting the interval into $[0, \pi]$ and $[-\pi, 0]$, put $\cos\theta = y$. Then

$$T_{l,m}(\theta) = \left\{ \begin{array}{c} \sum\limits_0^l A_k y^k + \sum\limits_0^{m-1} B_k y^k \sqrt{1-y^2} \\ \sum\limits_0^l A_k y^k - \sum\limits_0^{m-1} B_k y^k \sqrt{1-y^2} \end{array} \right\} =$$

$$= Q_l(y) \pm P_{m-1}(y)\sqrt{1-y^2} = \Phi_{1,2}(y).$$

2) If

$$\Phi_{1,2}(y) = Q_l(y) \pm P_{m-1}(y)\sqrt{1-y^2}$$

are normalized on $[-1, +1]$, then the substitution $y = \cos\theta$ in each of them $(0 \le \theta \le \pi)$ gives on $[-\pi, +\pi]$ a single normalized trigonometric polynomial.

Thus the question of constructing all $T_{l,m}(\theta)$ is equivalent to the question of finding all pairs $\Phi_{1,2}(y)$.

Remark 1. Under the indicated substitution (direct and inverse), the degrees l and m are preserved.

Remark 2. By the nodes of the polynomial $T_{l,m}(\theta)$ we shall mean the points θ_1^+, if

$$T_{l,m}(\theta_1^+) = +1,$$

and $\bar{\theta}_2$, if

$$T_{l,m}(\bar{\theta}_2) = -1.$$

If $(\bar{\theta}_k^{\pm(1)})$ are the nodes of $T_{l,m}$ on $[0, \pi]$, and $(\bar{\theta}_k^{\pm(2)})$ are its nodes on $[-\pi, 0]$, then $(\bar{\sigma}_k^{\pm(1)}) = (\cos\bar{\theta}_k^{\pm(1)})$ and $(\bar{\sigma}_k^{\pm(2)}) = (\cos\bar{\theta}_k^{\pm(2)})$ are, respectively, the nodes of $\Phi_1(y)$ and $\Phi_2(y)$ on $[-1, +1]$ (for Φ_1 the nodes are arranged in the reverse order).

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Extremal analytic functions and their properties

Remark on divisibility. If $f(y)$ is represented by its Maclaurin series $\sum_0^\infty a_i y^i$ on the whole interval $[-1,+1]$, then the "remainder upon division" of $f(y)$ by

$$R_s^2(y) = \prod_1^s (y-\sigma_i)^2,$$

where $-1 < \sigma_i < +1$, is the uniquely determined polynomial $r(y)$ of degree lower than $2s$ (see (2)). Since

$$\sqrt{1-y^2} = \sum_0^\infty (-1)^n \binom{1/2}{n} y^{2n}$$

converges absolutely for $-1 \leq y \leq 1$, $P(y)\sqrt{1-y^2}$ also gives a completely determined remainder upon division by $R_s^2(y)$.

By a distribution $(\overset{\pm}{\sigma}_i)_1^s$ of a polynomial (or function) we shall mean the complete set of its nodes, taking into account the sign assumed by the polynomial at this node (the polynomial or function is reduced).

Let some distribution $(\overset{\pm}{\sigma}_i)_1^s$ on $(-1,+1)$ be given ([2]), and let

$$R_1(x) = \prod_1^{s_1} (x - \overset{+}{\sigma}_i); \qquad R_2(x) = \prod_1^{s_2} (x - \overset{-}{\sigma}_i) \quad (s_1 + s_2 = s).$$

In the identity

$$R_1^2(x)\varphi(x) + R_2^2(x)\psi(x) \equiv 2$$

the polynomials $\varphi(x)$ and $\psi(x)$ are found uniquely by successive division, and their degrees are not higher than $2s_2 - 1$ and $2s_1 - 1$, respectively. A polynomial (of Hermite) of the form

$$Q_n(x) = 1 - R_1^2(x)\varphi(x) \equiv -1 + R_2^2(x)\psi(x)$$

will be the principal extremal polynomial of the distribution $(\sigma_i)_1^s$ (i.e. of the least degree) if and only if on $[0,1]$ one has $\varphi(x) \geq 0$ and $\psi(x) \geq 0$. Otherwise we shall call it quasi-extremal and denote it by $\widetilde{Q}_n(x)$ (the same on the interval $(-1,+1)$ after the substitution $y = 2x - 1$).

Remark 3. We shall call any analytic function $\Phi(y)$ extremal for the distribution $(\overset{\pm}{\sigma}_i)_1^s$ on $[-1,+1]$ if $\Phi(\sigma) = \pm 1$ and $\max_{[0,1]} |\Phi(y)| = 1$. $\Phi(y)$ is obtained from $\widetilde{Q}_n(y)$ by the formula

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$$\Phi(y) = \widetilde{Q}_n(y) + R_s^2(y)\Omega(y),$$

where $\Omega(y)$ is an analytic function and

$$-\psi(y)/R_1^2(y) \leq \Omega(y) \leq \varphi(y)/R_2^2(y). \tag{1}$$

The width of the "reducedness strip" is equal to $2/R_s^2(y)$ ($R_s = R_1 R_2$). In general, every extremal $\Psi(y)$ can be obtained from any other extremal $\Phi(y)$ by the formula $\Psi(y) = \Phi(y) + R_s^2(y)\overline{\Omega}(y)$, with the corresponding reducedness strip for $\overline{\Omega}(y)$.

Theorem 2. *If the given function reduced on $[-1, +1]$, with distribution $(\overset{\pm}{\sigma}_1)_1^s$, has the form*

$$\Phi(y) = (Q_l(y) + P_{m-1}(y)\sqrt{1-y^2}),$$

where $l < 2s$, then the identity

$$Q_l(y) \equiv \widetilde{Q}_n(y) - \rho(y), \tag{2}$$

holds, where $\rho(y)$ is the remainder upon division of $P_{m-1}(y)\sqrt{1-y^2}$ by $R_s^2(y)$.

Indeed,

$$P(y)\sqrt{1-y^2} \equiv \rho(y) + R_s^2(y)\Omega(y),$$

and, since $\Phi(y)$ is an extremal function, we have

$$\Phi(y) = \widetilde{Q}_n(y) + R_s^2(y)\Omega'(y) = Q_l(y) + \rho(y) + R_s^2(y)\Omega(y).$$

Hence, from the uniqueness both of the remainder and of the incomplete quotient, we obtain

$$\widetilde{Q}_n(y) = Q_l(y) + \rho(y), \qquad \Omega'(y) = \Omega(y).$$

Remark 4. In finding $\rho_{2s-1}(y)$ in the identity

$$P_{m-1}(y)\sqrt{1-y^2} = \rho_{2s-1}(y) + R_s^2(y)\Omega(y) \tag{3}$$

one may assume $m - 1 \leq 2s - 1$, since otherwise one can first find

$$P_{m-1}(y) = \rho(y) + R_s^2(y)\omega(y).$$

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Remark 5. In formula (3) the polynomials $P_{m-1}(y)$ and $\rho_{2s-1}(y)$ are determined uniquely from one another, being respectively Hermite polynomials: $\rho_{2s-1}(y)$ for the function $P_{m-1}(y)\sqrt{1-y^2}$, and $P_{m-1}(y)$ for the function $\rho_{2s-1}(y)/\sqrt{1-y^2}$.

Theorem 3. *If $T_{l,m}(\theta) = C_l(\theta) + S_m(\theta)\sqrt{1-y^2}$ is reduced to $[0,\pi]$ with distribution $\binom{\pm}{\theta_i}_1^s$ on $(0,\pi)$, where $0 < \theta_i < \pi$, then there exists an infinite set of trigonometric polynomials reduced to $[0,\pi]$ with the same distribution, for which: 1) $C_l(\theta)$ is preserved while the degree of $S_m(\theta)$ is raised; 2) $S_m(\theta)$ is preserved while the degree of $C_l(\theta)$ is raised.*

Indeed, pass from $T_{l,m}(\theta)$ to the corresponding

$$\Phi_1(y) = Q_l(y) + P_{m-1}(y)\sqrt{1-y^2}$$

with distribution $\binom{\pm}{\sigma}_1^s$ on $(-1,+1)$.

1) Construct, according to Remark 3, the extremal function

$$\Psi(y) = \Phi_1(y) + R_s^2(y)L_k(y),$$

where $L_k(y)$ is a polynomial (preferably of lowest possible degree). The reducedness requirement consists in the inequalities

$$-\Omega_2(y)/R_1^2(y) \leq L_k(y) \leq \Omega_1(y)/R_2^2(y), \qquad (4)$$

where

$$\Phi_1(y) = 1 - R_1^2(y)\Omega_1(y) \equiv -1 + R_2^2(y) - \Omega_2(y).$$

Then

$$\Psi(y) = [Q_l(y) + R_s^2(y)L_k(y)] + P_{m-1}(y)\sqrt{1-y^2}.$$

Returning to trigonometric form, we have

$$\Psi(\cos\theta) = C_{2s+k}(\theta) + S_m(\theta).$$

2) If the chosen $L_k(y)$ lies in the strip of reducedness (4), then $L_k(y)\sqrt{1-y^2}$ also lies in it, and we have

$$\Psi(y) = \Phi_1(y) + R_s^2(y)L_k(y)\sqrt{1-y^2} = Q_l(y) + [P_{m-1}(y) + R_s^2(y)L_k(y)]\sqrt{1-y^2}$$

—a function reduced on $[-1,+1]$ with distribution $\binom{\pm}{\sigma_i}_1^s$, after which we obtain

$$\Psi(\cos\theta) = C_l(\theta) + S_{2s+k}(\theta).$$

The theorem is proved.

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In what follows we shall be interested in trigonometric polynomials of possibly low degrees l and m with possibly large s.

Let $\binom{\pm}{\sigma_i}_1^s$, for $-1 < \sigma_i < 1$, be a complete distribution of a polynomial $Q_n(y)$ of class II, i.e. $n < 2s$. Then the quasi-extremal $\widetilde{Q}_n(y)$ coincides with $Q_n(y)$, and in the identity we have

$$\varphi(y) > 0, \qquad \psi(y) > 0$$

for $-1 \le y \le +1$.

Theorem 4. *One can always construct an infinite set of functions of the form*

$$\Phi_1(y) = Q_l(y) + P_{m-1}(y)\sqrt{1-y^2},$$

reduced to $[-1,+1]$*, having the complete distribution* $\binom{\pm}{\sigma_i}_1^s$ *of a given polynomial of class II* $Q_n(y)$*, where* $l < 2s$ *and* $m-1 < 2s$*; moreover, functions of the form*

$$\Phi_2(y) = Q_l(y) - P_{m-1}(y)\sqrt{1-y^2}$$

are also reduced to $[-1,+1]$ *(the distribution of* Φ_2 *remains unknown).*

Let us first write the conditions of double reducedness for Φ_1 and Φ_2 under the given distribution for $\Phi_1(y)$. By Theorem 2,

$$\Phi_{1,2}(y) = Q_n(y) - \rho(y) \pm P(y)\sqrt{1-y^2}.$$

It is required that

$$-1 \le Q_n(y) - \rho(y) \pm P(y)\sqrt{1-y^2} \le 1,$$

or

$$Q_n(y) - 1 \le \rho(y) \mp P(y)\sqrt{1-y^2} \le Q_n(y) + 1. \tag{5}$$

The inequalities (5) show that

$$Q_n(y) - 1 \le \rho(y) \le Q_n(y) + 1,$$

i.e. $\rho(y)$ must have no fewer sign changes than $Q_n(y)$. But since

$$P(\sigma_i)\sqrt{1-\sigma_i^2} = \rho(\sigma_i),$$

it follows that $P(y)$ also has no fewer sign changes.

If an arbitrarily chosen $\rho(y)$ of degree $< 2s$ inside the strip (5) and the $P(y)$ computed from it yield

$$\rho(y) \mp P(y)\sqrt{1-y^2}$$

outside the strip (5), replace $\rho(y)$ by $\lambda\rho(y)$ ($\lambda > 0$); then $P(y)$ is replaced by $\lambda P(y)$, and we have the same requirement for

$$\lambda\rho(y) \mp \lambda P(y)\sqrt{1-y^2}.$$

Write the conditions separately for Φ_1 (sign $-$) and Φ_2 (sign $+$):

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1)

$$Q_n(y) - 1 \le \lambda[\rho(y) - P(y)\sqrt{1-y^2}] \le Q_n(y) + 1$$

or

$$-R_1^2(y)\varphi(y) \le -\lambda R_s^2(y)\Omega(y) \le R_2^2(y)\psi(y),$$

which will always be satisfied for some

$$0 < \lambda \le \lambda_0;$$

2)

$$Q_n(y) - 1 \le \lambda[\rho(y) - P(y)\sqrt{1-y^2}] +$$

$$+2\lambda P(y)\sqrt{1-y^2} \le Q_n(y) + 1.$$

If 1) is satisfied, then λ can be decreased further so that 2) is also satisfied, since $P(y)\sqrt{1-y^2}$ is also inside the strip (5); the theorem is proved.

Corollary 1. Passing to trigonometry, we have: to each extremal polynomial $Q_n(y)$ of class II with distribution $(\bar{\sigma}_i^{\pm})_1^s$ on $(-1,+1)$ there corresponds an infinite set of trigonometric polynomials $T_{l,m}(\theta)$ possessing the following properties: $l < 2s,\ m < 2s$; $T_{l,m}(\theta)$ reduced on $[-\pi,+\pi]$; on $[0,\pi]$ it has $(\bar{\theta}_i^{\pm})_1^s$ nodes corresponding to the substitution $(\bar{\sigma}_i^{\pm}) = \cos\theta_i$.

Corollary 2. Each $Q_n(y)$ of class II generates a family $\{\Phi_{1,2}(y)\}_{Q_n(y)}$, containing $2s$ parameters—the coefficients of $\rho(y)$, independent but bounded by some $\lambda_{\max}$.

Corollary 3. Any analytic pair $\Phi_{1,2}(y)$ (generating $T_{l,m}(\theta)$) with the distribution of at least one of them, say $\Phi_1(y)$, of class II belongs to $\{\Phi_{1,2}(y)\}_{Q_n(y)}$, where $Q_n(y)$ is the principal polynomial of this distribution. In particular, this includes all such pairs for which $\Phi_2(y)$ also has on $[-1,+1]$ some distribution of class II $(\bar{\sigma}_i^{\pm})_1^{s'}$; we denote such pairs by $\Phi_{1,2}^*(y)$ and call them analytic pairs of class II.

Corollary 4. If there is a pair $\Phi_{1,2}^*(y)$ with corresponding distributions $(\bar{\sigma}_i^{\pm})_1^{s_1}$ and $(\bar{\sigma}_i^{\pm})_1^{s_2}$ and extremal polynomials $Q_{n_1}(y)$ and $Q_{n_2}(y)$, then this pair is contained in both families $\{\Phi_{1,2}(y)\}_{Q_{n_1}}$ and $\{\Phi_{1,2}(y)\}_{Q_{n_2}}$. Conversely: if two such families have a common pair, then it is $\Phi_{1,2}^*(y)$.

Theorem 5 (criterion for analytic pairing of class II). Necessary and sufficient conditions for given $Q_{n_1}(y)$ and $Q_{n_2}(y)$ with their complete distributions $(\bar{\sigma}_i^{\pm})_1^{s_1}$ and $(\bar{\sigma}_i^{\pm})_1^{s_2}$ to generate a pair of class II

$$\Phi_{1,2}^*(y) = Q_1(y) \pm P_{m-1}(y)\sqrt{1-y^2}$$

are the following: there exists a polynomial $P(y)$ of degree $< 2s',\ 2s''$, possessing the following properties: if $\rho_1(y)$ and $\rho_2(y)$ are the remainders, respectively, from

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division of $P(y)\sqrt{1-y^2}$ by $R^2_{s'}(y)$ and of $-P(y)\sqrt{1-y^2}$ by $R^2_{s''}(y)$, then the conditions

$$Q_{n_2}(y) - Q_{n_1}(y) \equiv \rho_2(y) - \rho_1(y); \tag{6}$$

$$Q_{n_1}(y) - 1 \leq \rho_1(y) \mp P(y)\sqrt{1-y^2} \leq Q_n(y) + 1. \tag{7}$$

are satisfied.

On the basis of what has been set forth above, these conditions require no proof, being evident. The inequalities (7) are preserved, in consequence of (6), when n_1 is replaced by n_2 and $\rho_1(y)$ by $\rho_2(y)$.

Remark 6. The problem of actually checking the suitability of $Q_{n_1}(y)$ and $Q_{n_2}(y)$ for forming a pair $\Phi^*_{1,2}(y)$ is a linear problem. Indeed, taking

$$P(y) = \sum_0^{2s-1} a_i y^i$$

with undetermined coefficients ($s = \max s'$ and s''), we find unique

$$\rho_1(y) = \sum_1^{2s_1-1} b_i^{(1)} y^i \quad \text{and} \quad \rho_2(y) = \sum_1^{2s_2-1} b_i^{(2)} y^i,$$

where $(b_i^{(1)})$ and $(b_i^{(2)})$ are linear homogeneous functions of (a_i). The identity (6) decomposes into $2s$ equations with unknowns $(a_i)_0^{2s-1}$, which, generally speaking, have unique solutions, after which (7) is checked.

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1. E. V. Voronovskaya, *DAN*, **129**, No. 1 (1959).

2. E. V. Voronovskaya, *The Method of Functionals and Its Applications*, L., 1963.

Note: Figure translations are in progress. See original paper for figures.

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