SOLUTION OF A NONLINEAR PROBLEM ON NONSTATIONARY ATMOSPHERIC MOTIONS OF PLANETARY SCALE

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Abstract

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For solving the nonstationary problem of atmospheric motions of planetary scale, the equations may serve ((1), the baroclinic two-level problem)

$$L_1 \equiv \frac{\partial \Delta\Psi}{\partial t} + \frac{1}{a_0^2 \sin\theta}\left[(\Psi, \Delta\Psi) + (\psi, \Delta\psi)\right] + 2\omega\frac{\partial\Psi}{\partial\lambda} = 0, \tag{1}$$

$$L_2 \equiv \frac{\partial \Delta\psi}{\partial t} - \frac{2.5}{\Gamma}\frac{\partial\psi}{\partial t} + \frac{1}{a_0^2 \sin\theta}\left[(\Psi, \Delta\psi) + (\psi, \Delta\Psi) + \frac{2.5}{\Gamma}(\psi, \Psi)\right] + 2\omega\frac{\partial\psi}{\partial\lambda} = 0. \tag{2}$$

Here $\Psi = \frac{1}{2}(\Psi_1 + \Psi_3)$; $\psi = \frac{1}{2}(\Psi_1 - \Psi_3)$; Ψ_1 and Ψ_3 are the stream functions on the 300 and 700 mb surfaces, respectively; a_0 and ω are the radius and angular velocity of rotation of the Earth; t is time; θ and λ are spherical coordinates (θ is the complement of latitude, λ is longitude of the place);

$$(M, N) = \frac{\partial M}{\partial\theta}\frac{\partial N}{\partial\lambda} - \frac{\partial M}{\partial\lambda}\frac{\partial N}{\partial\theta}; \qquad \Gamma = \frac{R^2 T_1}{4g\omega^2 a_0^2}\left(\frac{\gamma_a - \gamma}{\cos^2\theta}\right)_{\text{cp}} = \text{const (see (1))}.$$

We seek the solution in the form:

$$-\frac{1}{a_0^2\omega}\Psi = -\bar{a}(\tau)P_1(\cos\theta) + \left[H(\tau)\cos m\lambda + H'(\tau)\sin m\lambda\right]P_n^m(\cos\theta), \tag{3}$$

$$-\frac{1}{a_0^2\omega}\psi = -\delta(\tau)P_1(\cos\theta) + \left[h(\tau)\cos m\lambda + h'(\tau)\sin m\lambda\right]P_n^m(\cos\theta), \tag{4}$$

where $\tau = \omega t$ (dimensionless time); $\bar{a}, \delta, H, H', h, h'$ are unknown functions of time (with initial values $a_0, \delta_0, H_0, H_0', h_0, h_0'$, respectively); n, m are integers ($n > m$, $n - m$ odd); P_1 and P_n^m are Legendre polynomials.

With the aid of (3), (4) we shall try to satisfy 6 equations

$$\int_0^{2\pi}\int_0^{\pi/2} L_1 V_i \sin\theta\, d\theta\, d\lambda = 0, \qquad \int_0^{2\pi}\int_0^{\pi/2} L_2 V_i \sin\theta\, d\theta\, d\lambda = 0 \quad (i = 1, 2, 3), \tag{5}$$

where we take $V_1 = P_1,\ V_2 = \cos m\lambda\, P_n^m,\ V_3 = \sin m\lambda\, P_n^m$.

Substituting (3) and (4) into (5), we obtain, for $i = 1$:

$$d\bar{a}/d\tau = 0, \tag{6}$$

$$d\delta/d\tau = C(Hh' - H'h), \tag{7}$$

where

$$C = \frac{1}{1 + 0.8\,\Gamma}\,\frac{3m}{2(2n+1)}\,\frac{(n+m)!}{(n-m)!}.$$

For $i = 2, 3$ we shall have

$$dH/d\tau = AH' - a\delta h', \tag{8}$$

$$dH'/d\tau = -AH + a\delta h, \tag{9}$$

$$dh/d\tau = Bh' - b\delta H', \tag{10}$$

$$dh'/d\tau = -Bh + b\delta H, \tag{11}$$

where

$$A = \frac{m}{n(n+1)}\{2 - [n(n+1) - 2]\bar{a}\}, \qquad a = m\,\frac{n(n+1) - 2}{n(n+1)},$$

$$B = \frac{m}{n(n+1) + 2.5/\Gamma}\left\{2 - \left[n(n+1) - 2 + \frac{2.5}{\Gamma}\right]\bar{a}\right\},$$

$$b = m\,\frac{n(n+1) - 2 - 2.5/\Gamma}{n(n+1) + 2.5/\Gamma}.$$

The system of nonlinear equations (6)–(11) can be solved in closed form. Let us show this. The first integral is obtained immediately from (6):

$$\bar{a} = a_0 = \text{const}. \qquad (12)$$

Multiplying (8) by H, (9) by H', adding the resulting equations, replacing $Hh' - H'h$ according to (7), and carrying out the quadratures, we find the second integral:

$$H^2 + {H'}^2 + \frac{a}{C}\delta^2 = H_0^2 + H_0'^2 + \frac{a}{C}\delta_0^2 = C_1. \qquad (13)$$

Multiplying (10) by h, (11) by h', adding the resulting equations and again using (7), we find the third integral:

$$h^2 + {h'}^2 - \frac{b}{C}\delta^2 = h_0^2 + h_0'^2 - \frac{b}{C}\delta_0^2 = C_2. \qquad (14)$$

Further, multiplying (8) by h, (9) by h', (10) by H, (11) by H', adding the resulting equations and eliminating $Hh' - H'h$ according to (7), we find the fourth integral:

$$Hh + H'h' + \frac{A-B}{C}\delta = H_0h_0 + H_0'h_0' + \frac{A-B}{C}\delta_0 = C_3. \qquad (15)$$

Let us write $\frac{d}{d\tau}(hH' - Hh')$, using the expressions for the derivatives from (8)–(11) and then eliminating $H^2 + {H'}^2$, $h^2 + {h'}^2$, $Hh + H'h'$ according to (13)–(15):

$$\frac{d}{d\tau}(hH' - Hh') = \frac{2ab}{C}\delta^3 + \left[\frac{(A-B)^2}{C} + aC_2 - bC_1\right]\delta - (A-B)C_3. \qquad (16)$$

Substituting this expression for the derivative into equation (7), differentiated with respect to τ, we obtain, for determining δ,

$$d^2\delta/d\tau^2 = -2ab\delta^3 - \left[(A-B)^2 + C(aC_2 - bC_1)\right]\delta + (A-B)CC_3. \qquad (17)$$

Thus,

Fig. 1

Figure 1: Fig. 1

$$\tau = \pm \int_{\delta_0}^{\delta} \frac{d\delta}{\sqrt{S(\delta)}}, \tag{18}$$

where $S(\delta)$ is a fourth-degree polynomial in δ:

$$S(\delta) \equiv -ab\delta^4 - \left[(A-B)^2 + C(aC_2 - bC_1)\right]\delta^2 + 2(A-B)CC_3\delta + 2C_4, \tag{19}$$

with

$$2C_4 \equiv ab\delta_0^4 + \left[(A-B)^2 + C(aC_2 - bC_1)\right]\delta_0^2 - 2(A-B)CC_3\delta_0 + C^2(H_0h_0' - H_0'h_0)^2.$$

We now introduce the amplitudes and phases of the waves under consideration. Let

$$H = R(\tau)\cos E(\tau), \qquad H' = R(\tau)\sin E(\tau),$$

$$h = \rho(\tau)\cos e(\tau), \qquad h' = \rho(\tau)\sin e(\tau). \tag{20}$$

Then from (13) and (14) one can write:

$$R^2 = C_1 - \frac{a}{C}\delta^2, \tag{21}$$

$$\rho^2 = C_2 + \frac{b}{C}\delta^2. \tag{22}$$

Thus, R and ρ are determined through δ.

Further, from (20) it follows that $hH + h'H' = \rho R\cos(E-e)$. Replacing the left-hand side of this equality by (15), we obtain

$$R\rho\cos(E-e) = C_3 - \frac{A-B}{C}\delta. \tag{23}$$

Fig. 1

It remains for us to determine E or e.

Multiplying (8) by H', subtracting from the result equation (9) multiplied by H, and using here (20), (15), and (21), we obtain

$$\frac{dE}{d\tau} = -B + \frac{a}{R^2}\left(C_3\delta - \frac{A-B}{a}C_1\right). \tag{24}$$

The right-hand side of (24) is a known function of τ (or δ). It remains to carry out here a quadrature with respect to time*.

If the quantities H, H', h, h' are small, so that their quadratic terms may be neglected, then by (7) $\delta = \delta_0 = \text{const}$, and the system (8)–(11) will be a system of linear equations. This case was studied, in particular, in work (2), where it was shown that for certain values of n and δ_0 the motion proves to be unstable. In the nonlinear case the picture of the motion will be essentially different. The principal property of the solution we have obtained for the nonlinear problem is the variation of δ with time. On the basis of the analysis of (18) we may speak of oscillations of δ with a definite period, which depends on the initial data and which can in each case be found in advance. Thus, for example, for

$$n = 7, \quad m = 4, \quad 2.5/\Gamma = 104, \quad \alpha_0 = 0.04,$$

$$\delta_0 = 0.025$$

(to these parameter values there corresponds an unstable motion in the linear problem),

$$\sqrt{C}H_0 = 4 \cdot 10^{-2}, \quad \sqrt{C}H_0' = 3 \cdot 10^{-2},$$

$$\sqrt{C}h_0 = 3 \cdot 10^{-2}, \quad h_0' = 0$$

we obtain, as the roots of the equation $S(\delta) = 0$:

$$\delta_1 \approx 0.037, \quad \delta_2 \approx 0.034, \quad \delta_3 \approx 0.010, \quad \delta_4 \approx -0.081.$$

The initial value of δ lies between δ_2 and δ_3, and within these limits $\delta(\tau)$ will undergo oscillations. With the aid of tables of complete elliptic integrals (for example, (3)), it is not difficult to compute also the period T of these oscillations. In our example it turned out that $T \approx 13.5$ days. In Figs. 1 and 2, for this example, the curves are given in the phase planes (H, H'), (h, h'), respectively. The numbers assigned to the points on the curves denote the time in days. The solution of the nonlinear problem is stable.

Fig. 2

Fig. 2

Figure 2: Fig. 2

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* In the concrete calculations we passed from the variable τ to the variable δ. In doing so, E was expressed through δ in the form of a combination of elliptic integrals of the 1st and 3rd kinds.

Note: Figure translations are in progress. See original paper for figures.