Reports of the Academy of Sciences of the USSR

1968

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1968. Vol. 179, No. 5

MATHEMATICS

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ON THE APPROXIMATE SOLUTION OF SOME INFINITE SYSTEMS OF EQUATIONS

(Presented by Academician I. N. Vekua on 6 VI 1967)

Variational methods for determining the coefficients a_j of an expansion of the solution of functional equations in some system $\{\varphi_j\}$ lead (1) to the following infinite system of equations:

$$\sum_{j=1}^{\infty}(A_1\varphi_j, A_2\varphi_k)a_j = (A_3\varphi, A_4\varphi_k) \quad (k = 1, 2, \dots), \tag{1}$$

where A_i $(i = 1, 2, 3, 4)$ are positive or positive-definite (1) operators, and φ is a certain known function. If the sequences $\{A_1\varphi_j\}$ and $\{A_2\varphi_j\}$ are biorthonormal, or if $A_1 \equiv A_2$ the sequence $\{A_1\varphi_j\}$ is orthonormal, then for the coefficients a_j we obtain

$$a_j = (A_3\varphi, A_4\varphi_k). \tag{2}$$

In some cases, however, the system $\{A_1\varphi_j\}$ is not strongly minimal, and its preliminary orthonormalization encounters great difficulties (3). Therefore the question arises of the approximate solution of system (1). By an approximate solution of system (1) we shall mean the vector $\bar{a}^{(N)}(a_1^{(N)}, \dots, a_N^{(N)})$, which satisfies the system

$$A^{(N)}\bar{a}^{(N)} = B^{(N)} + \varepsilon \tag{3}$$

or

$$\sum_{j=1}^{N}(A_1\varphi_j, A_2\varphi_k)a_j^{(N)} = (A_3\varphi, A_4\varphi_k) + \varepsilon_k \quad (k = 1, 2, \dots, N),$$

where $\varepsilon(\varepsilon_1, \dots, \varepsilon_N)$ is the residual vector. For the vector $(a^{(N)} - \bar{a}^{(N)})$ to be small, where $a^{(N)}$ is the solution of system (3) for $\varepsilon \equiv 0$ ($a_i^{(N)}$ are the coefficients of the best, in a certain sense, expansion of the solution of the functional equation in the system $\{\varphi_i\}$), it is necessary that the expression $\|(A^{(N)})^{-1}\varepsilon\|$ be small. However, in the case of unreliable systems (which are not a Bari basis ([4])), the norm of the vector $(a^{(N)} - \bar{a}^{(N)})$ may be arbitrarily large and, nevertheless, the difference between the function ψ being expanded and its approximation $\sum_{i=1}^{N} a_i^{(N)} \varphi_i$ may, in some metric, be less than any sufficiently small number $\varepsilon > 0$:

$$\left\| \psi - \sum_{i=1}^{N} a_i^{(N)} \varphi_i \right\| < \varepsilon. \tag{4}$$

In the latter case the corresponding matrix is ill-conditioned, and in its neighborhood there is a singular matrix. As shown in ([5]), without regularization we may then obtain strongly differing solutions, distinct from the normal solution.

Therefore, by an ε-approximate solution of system (1) we shall call a vector $\bar{a}^{(N)}$ satisfying inequality (4).

In the present note we indicate one new method for obtaining approximate solutions of the system (1). The essence of the new method is that, in the case of certain systems $\{\varphi_i\}$ (the corresponding conditions for the systems will be formulated below), in order to obtain an approximate solution of the system (1) for large N, it is sufficient to carry out a single Seidel iteration, taking the zero vector as the initial approximation.

Let there be a Hilbert space H, in which the scalar product $[u, v]$ is defined as follows:

$$[u, v] = (A_1 u, A_2 v) = (A_1 v, A_2 u),$$

and let the best approximation, in the metric H, to the function ψ be sought by a generalized polynomial $\sum_{j=1}^{N} a_j \varphi_j$. From the orthogonality of the difference $\left(\psi - \sum_{j=1}^{n} a_j \varphi_j\right)$ to an arbitrary function φ_j, we obtain, for determining the coefficients a_j, the system

$$\left[\psi - \sum_{j=1}^{N} a_j \varphi_j, \varphi_k\right] = 0, \qquad k = 1, 2, \dots, N,$$

or

$$\sum_{j=1}^{N} a_j (A_1 \varphi_j, A_2 \varphi_k) = (A_1 \psi, A_2 \varphi_k).$$

If the operator A_1 is the product $A_1 = A_5 A_4 A_6$, where A_5 satisfies the condition

$$(A_5 A_4 \varphi, A_2 \varphi_i) = (A_4 \varphi, A_5 A_2 \varphi_i),$$

and the functional equation

$$A_6 \psi = \varphi$$

is given, then, denoting $A_5 A_2 = A_3$, for determining the coefficients a_j we obtain the system

$$\sum_{j=1}^{N} a_j (A_1 \varphi_j, A_2 \varphi_k) = (A_3 \varphi_k, A_4 \varphi).$$

We shall determine approximate values $\bar{a}_j$ of the coefficients a_j by means of the Seidel iteration process, taking as the initial approximation the vector $a^0(0, \ldots, 0)$,

$$\bar{a}_1 = \frac{(A_3 \varphi_1, A_4 \varphi)}{(A_1 \varphi_1, A_2 \varphi_1)}, \quad \bar{a}_2 = \frac{(A_3 \varphi_2, A_4 \varphi) - \bar{a}_1 (A_1 \varphi_1, A_2 \varphi_2)}{(A_1 \varphi_2, A_2 \varphi_2)}, \ldots$$

$$\ldots, \bar{a}_k = \left[(A_3 \varphi_k, A_4 \varphi) - \sum_{j=1}^{k-1} \bar{a}_j (A_1 \varphi_j, A_2 \varphi_k) \right] \Big/ (A_1 \varphi_k, A_2 \varphi_k).$$

Let us note that if the systems $\{A_1 \varphi_j\}$, $\{A_2 \varphi_j\}$ are biorthonormal, then the coefficients $\bar{a}_j$ coincide with the coefficients (2) of the function ψ. Since

$$(A_1 \psi, A_2 \varphi_k) = (A_3 \varphi_k, A_4 \varphi).$$

then for $\bar{a}_k$ we obtain

$$\bar{a}_k = \left(A_1 \left(\psi - \sum_{j=1}^{k-1} \bar{a}_j \varphi_j \right), A_2 \varphi_k \right) \Big/ (A_1 \varphi_k, A_2 \varphi_k).$$

Thus, $\bar{a}_k$ is the coefficient of the best approximation, in the sense of the Hilbert space H under consideration, of the difference $(\psi-$

$$- \sum_{j=1}^{k-1} \bar{a}_j \varphi_j) = \varphi^{(k-1)}$$

by the function $c_k \varphi_k$ (6)

$$\min_{c_k}\left\|\psi-\sum_{j=1}^{k-1}\bar{a}_j\varphi_j-c_k\psi_k\right\|=\left\|\varphi-\sum_{j=1}^{k}\bar{a}_j\varphi_j\right\|=\frac{G(\varphi^{(k-1)},\varphi_k)}{G(\varphi_k)},\tag{5}$$

where $G(u_1,\dots,u_n)$ is the Gram determinant of the functions $u_1,\dots,u_n$. Since

$$G(\varphi^{(k-1)},\varphi_k)/G(\varphi_k)=\left(\|\varphi^{(k-1)}\|\right)^2-\left(A_1\varphi^{(k-1)},A_2\varphi_k\right)^2/\|\varphi_k\|^2,$$

from expression (5) we obtain

$$\left(\|\varphi^{(k)}\|\right)^2=\left(\|\varphi^{(k-1)}\|\right)^2-\left(A_1\varphi^{(k-1)},A_2\varphi_k\right)^2/\|\varphi_k\|^2.\tag{6}$$

Thus, the sequence of positive numbers $\{\|\varphi^{(k)}\|\}$ is monotonically decreasing and, consequently, has a limit. Assuming that the norms of the functions φ_i are bounded in the aggregate both above and below, from (6) we obtain that for any $\varepsilon>0$ there is an N_0 such that

$$\sum_{k=N_0}^{\infty}\left(A_1\varphi^{(k-1)},A_2\varphi_k\right)^2<\varepsilon.\tag{7}$$

But then, for $s>N_0$,

$$\begin{aligned}\varepsilon&>\left|\left(\|\varphi^{(s)}\|\right)^2-\left(\|\varphi^{(s+1)}\|\right)^2\right|\\&=\left|\left(A_1\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right),A_2\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right)\right)\right.\\&\qquad\left.-\left(A_1\left(\psi-\sum_{j=1}^{s+1}a_j\varphi_j\right),A_2\left(\psi-\sum_{j=1}^{s+1}a_j\varphi_j\right)\right)\right|\\&=\left|\left(A_1\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right),A_2\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right)\right)\right.\\&\qquad-2a_{s+1}\left(A_1\varphi_{s+1},A_2\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right)\right)-a_{s+1}^2\left(A_1\varphi_{s+1},A_2\varphi_{s+1}\right)\\&\qquad\left.-\left(A_1\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right),A_2\left(\psi-\sum_{j=1}^{s}a_j\varphi_j\right)\right)\right|\\&=\left|-3a_{s+1}^2\left(A_1\varphi_{s+1},A_2\varphi_{s+1}\right)\right|,\end{aligned}$$

and, by virtue of the boundedness of the norm of the functions φ_j from below,

$$|a_{s+1}| < \varepsilon.$$

From the last inequality we obtain that, for any $\varepsilon > 0$ and integer N, there is an N_0 such that

$$\varphi^{(s)} = \varphi^{(r)} + \gamma_r^{(s)} \quad (N_0 \leq s,\ r \leq N_0 + N), \tag{8}$$

where

$$\|\gamma_r^{(s)}\| < \varepsilon. \tag{9}$$

Substituting (8) into (7), we obtain

$$\begin{aligned}\varepsilon > \sum_{k=N_0}^{\infty} \left(A_1\varphi^{(k-1)}, A_2\varphi_k\right)^2 &\geq \sum_{k=N_0}^{N_0+N} \left(A_1\varphi^{(k)}, A_2\varphi_{k-1}\right)^2 \\ = \sum_{k=N_0}^{N_0+N} \left(A_1\varphi^{(r)}, A_2\varphi_{k-1}\right)^2 &+ \sum_{k=N_0}^{N_0+N} \left(A_1\gamma_r^{(k)}, A_2\varphi_{k-1}\right)^2 \\ + 2\sum_{k=N_0}^{N_0+N} (A_1\varphi^{(r)}, A_2\varphi_{k-1})&(A_1\gamma_r^{(k)}, A_2\varphi_{k-1}).\end{aligned}$$

But, taking (9) into account, we find that for any $\varepsilon > 0$ and integer N there exists such an N_0 that

$$\sum_{k=N_0}^{N_0+N} (A_1\varphi^{(r)},\ A_2\varphi_{k-1})^2 < \varepsilon \qquad (N_0 \leq r \leq N_0 + N). \tag{10}$$

We shall assume that the system $\{\varphi_i\}$ satisfies the following condition: for any $\varepsilon > 0$, integer N, and $\psi \in H$, there exist coefficients b_k $(k = N_0, \dots, N_0 + N)$ such that, for at least one value of r,

$$\left\|\psi^{(r)} - \sum_{k=N_0}^{N_0+N} b_k\varphi_k\right\| < \varepsilon \qquad (N_0 \leq r \leq N_0 + N). \tag{11}$$

$$\sum_{k=N_0}^{N_0+N} b_k^2 < M, \tag{12}$$

where M is a constant independent of N_0 and N.

By virtue of (10), (11), (12), and the Schwarz–Bunyakovsky inequality, we obtain

$$\left|\sum_{k=N_0}^{N_0+N} b_{k-1}(A_1\varphi^{(r)},\, A_2\varphi_{k-1})\right| \ll \varepsilon_1.$$

Taking into account the last two inequalities and the fact that $\|\varphi^{(r)}\| \leq \|\psi\|$ $(r = 1, 2, \ldots)$, we obtain

$$\left\|\psi - \sum_{j=1}^{r} \bar{a}_j\varphi_j\right\|^2 = \|\varphi^{(r)}\|^2 = \left(A_1\varphi^{(r)},\, A_2\sum_{k=N_0}^{N_0+N} b_k\varphi_k\right)$$
$$- \left(A_1\varphi^{(r)},\, A_2\left(\sum_{k=N_0}^{N_0+N} b_k\varphi_k - \varphi^{(r)}\right)\right) \ll$$
$$\ll \varepsilon_1 + \|\varphi^{(r)}\|\left\|\varphi^{(r)} - \sum_{k=N_0}^{N_0+N} b_k\varphi_k\right\| \ll \varepsilon_1 + \|\psi\|\varepsilon \ll \varepsilon_2,$$

where ε_2 is an arbitrarily small number.

Thus, the following theorem has been proved.

If the norms of the functions φ_j *in the Hilbert space* H *are bounded both above and below, the system* $\{\varphi_j\}$ *and the function* ψ *satisfy conditions (11) and (12), then, for an* ε*-approximate solution of system (1), it is sufficient to carry out a single Zeidel iteration for the system*

$$\sum_{j=1}^{N(\varepsilon)}(A_1\varphi_j,\, A_2\varphi_k)a_j = (A_3\varphi_k,\, A_4\varphi) \qquad (k = 1, 2, \ldots, N(\varepsilon)),$$

taking as the initial approximation the zero vector $a^0(0, 0, \ldots, 0)$.

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Received
1 VI 1967

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Note: Figure translations are in progress. See original paper for figures.