SPECTRAL ANALYSIS OF UNBOUNDED NON-SELF-ADJOINT OPERATORS IN A SPACE WITH INDEFINITE METRIC

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.432

MATHEMATICS

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SPECTRAL ANALYSIS OF UNBOUNDED NON-SELF-ADJOINT OPERATORS IN A SPACE WITH INDEFINITE METRIC

(Presented by Academician L. S. Pontryagin on 10 III 1967)

In this communication, by the method of characteristic matrix functions, a broad class of unbounded non-self-adjoint operators acting in a space with indefinite metric is investigated. In the case of bounded non-self-adjoint or nonunitary operators, analogous results were obtained in the communications ($^{1-3}$).

1. Let H be a Hilbert space; $(\cdot,\cdot)$ the scalar product in H; P a bounded self-adjoint continuously invertible operator acting in the space H. We introduce a metric in H by putting $[f,g]=(Pf,g)$. The space H with scalar product $[\cdot,\cdot]$ is, generally speaking, a space with indefinite metric, which we shall denote by Π, and we shall speak respectively of the H-metric (Hilbert metric) and the Π-metric (indefinite metric).

A subspace of the space Π is called, as usual, the linear envelope, closed in the H-metric, of some system of vectors from Π. In what follows, an important role is played by projectively complete subspaces, which were introduced independently in the works of E. Shaybe (^{4}) and I. S. Iokhvidov (^{5}) and studied in detail by Yu. P. Ginzburg and I. S. Iokhvidov (^{6}). For us it is important that if Π_1 is a projectively complete subspace of the space Π, then the subspace* $\Pi_2=\Pi[-]\Pi_1$ is also projectively complete, and moreover $\Pi=\Pi_1[+]\Pi_2$.

2. By analogy with the definite case (^{7}), a linear closed operator A with dense domain D_A in Π will be called an N-operator. Denote by $\mathcal{G}_A$ the set of vectors $f\in D_A$ for which $[Af,g]=[f,Ag]$ for every $g\in D_A$. The manifold $\mathcal{G}_A$ is called the **domain of Hermiticity** of the operator A and is the broadest manifold on which $A=A^0$ (A^0 is the operator adjoint to A in the Π-metric).

Definition. An N-operator A is called a **quasi-Hermitian operator of rank** r (or a K^r-operator) if: 1) the set $\rho(A,A^0)$ of common regular points of the operators A and A^0 contains at least one nonreal point; 2) $\dim D_A = r \pmod{\mathcal{G}_A}$.

In this case the K^r-operator A is called: 1) a K^r_I-operator if $D_A=D_{A^0}$; 2) a K^r_{II}-operator if $\mathcal{G}_A=\Pi$; 3) a K^r_{III}-operator if $D_A\neq D_{A^0}$, $\mathcal{G}_A\neq\Pi$. Thus the set of K^r-operators is divided into three disjoint classes of operators. We note that the bounded operators studied in (3) belong to the class of K_I^r-operators.

3. Let $i \in \rho(A, A^0)$. Consider the auxiliary operator

$$B = iR_{-i} - iR_{-i}^0 - 2R_{-i}^0 R_{-i} \qquad (R_{-i} = (A + iI)^{-1}).$$

This operator maps the space Π into the defect subspace $\mathfrak{M}_{-i}$ of the operator A (which is defined by the equality $\mathfrak{M}_{-i} = \Pi[-]\mathfrak{M}_{-i}$, where $\mathfrak{M}_{-i} = (A + iI)D_A$) and can be represented in the form

$$B = \sum_{\alpha,\beta=1}^{s} [\cdot, h_\alpha] J_{\alpha\beta} h_\beta \quad (s \geq r),$$

where $J = \|J_{\alpha\beta}\|$ is a certain Hermitian and unitary matrix. In what follows, we shall call the collection of vectors $\{h_k\}_{k=1}^s$ an α-basis of the operator A, and the matrix J the corresponding coefficient matrix.

Let $\{h_k\}_{k=1}^s$ be an α-basis of the operator A, and let J be the corresponding coefficient matrix. Then the matrix

$$\tau = I - 2JG \quad (G = \|[h_k, h_i]\|) \tag{1}$$

is a nonsingular matrix and is called the distortion matrix of the operator A. Moreover, the matrix $(J\tau)^{-1}$ can be represented in the form

$$(J\tau)^{-1} = C^* J^{(0)} C, \tag{2}$$

where $J^{(0)}$ is a certain unitary and Hermitian matrix, and C is a nonsingular matrix depending on the choice of the matrix $J^{(0)}$.

We shall now agree to call the matrix $\chi_A(\lambda)$, defined by the relation

$$\chi_A(\lambda) J^{(0)} \chi_A^*(i) = J + i(\lambda + i)\|[(A^0 - iI)(A^0 - \lambda I)^{-1} h_h, h_i]\|, \tag{3}$$

the **characteristic matrix-function** (c.m.-f.) of the K^r-operator A. Setting $\lambda = i$ in (3) and using relations (1) and (2), we see that $\chi_A(i)$ is a certain nonsingular solution of the matrix equation $XJ^{(0)}X^* = J\tau$, and, consequently, the definition of the c.m.-f. is not contradictory.

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* Here and below, $[-]$ and $[+]$ denote, respectively, the signs of orthogonal complement and orthogonal sum in the Π-metric.

4. We shall call a subspace $\Pi_1 \subset \Pi$ **invariant** with respect to the operator A if $A(D_A \cap \Pi_1) \subset \Pi_1$ and $\overline{D_A \cap \Pi_1} = \Pi_1$. It is easy to verify that $A^0(D_{A^0} \cap \Pi_2) \subset \Pi_2$, where $\Pi_2 = \Pi[-]\Pi_1$. However, it may happen that $\overline{D_{A^0} \cap \Pi_2} \neq \Pi_2$. In that case, on the basis of the preceding, the subspace Π_2 is not invariant with respect to the operator A^0.

Definition. An operator A, acting in the space Π, is called a **coupling** of the operators A_1 and A_2, acting respectively in the subspaces Π_1 and $\Pi_2 = \Pi[-]\Pi_1$, if Π_1 and Π_2 are projection-complete invariant subspaces respectively with respect to the operators A and A^0, and, moreover,

$$A_1 = A|_{D_A \cap \Pi_1}, \qquad A_2 = \left[A^0|_{D_{A^0} \cap \Pi_2}\right]^0,$$

Theorem 1 (multiplication theorem). *Let the K^r-operator A be a coupling of a K^{r_1}-operator A_1 and a K^{r_2}-operator A_2 $(r_1, r_2 \leq r)$, and let $\overline{\lambda}$ be a common regular point of these operators. Then, in certain α-bases, the c.m.-f. of the operators A, A_1, and A_2 are connected with one another by the relation*

$$\chi_A(\lambda) = \chi_{A_1}(\lambda)\chi_{A_2}(\lambda).$$

5. We shall agree to call the largest invariant subspace Π_A of the operator A the **additional subspace** of this operator, if $D_A \cap \Pi_A \subset \mathcal{G}_A$ and in Π_A the operator A induces a self-adjoint operator. The operator A_p, which coincides with A on the manifold $D_A \cap (\Pi[-]\Pi_A)$, is called the **simple part** of the operator A. In the case when $A = A_p$, the operator A is called **simple**.

Theorem 2 (isomorphism criterion). *Let the c.m.-f. of the K^r-operators A and A' coincide. Then for $J = J'$ the simple parts of these operators are isomorphic, and for $J = -J'$ they are co-isomorphic* ([3]).

The preceding results make it possible to establish the following assertions:

Theorem 3. Let λ_0 be a nonreal $(+)$-proper ([3]) or $(-)$-proper value of the K^r-operator A. If, moreover, $\overline{\lambda}_0$ is a regular point of the operator A, then $\det \chi_A(\lambda_0) = 0$.

Theorem 4. If $\det \chi_A(\lambda_0) = 0$, then λ_0 is a point of the spectrum of the operator A.

Theorem 5. If the characteristic matrix function of a K^r-operator A, which acts in a nondegenerate space, is a constant matrix, then the operator A has a nontrivial invariant subspace that coincides with its isotropic part.

6. An operator A acting in the space Π will be called **dissipative** if for every $f \in D_A$, $\operatorname{Im}[Af, f] \geq 0$. It is easy to verify that the nonreal $(+)$-proper ($(-)$-proper) values of a dissipative operator A lie in the half-plane $\operatorname{Im}\lambda > 0$ ($\operatorname{Im}\lambda < 0$). The following assertions also hold:

I. The eigenvectors of a dissipative K^r-operator A corresponding to real or (0)-proper values belong to the domain of Hermiticity $\mathfrak{G}_A$ of the operator A.

II. If a simple dissipative K^r-operator A acts in a nondegenerate space, then it has no real or (0)-proper values.

7. Let now the K^r-operator A act in the space Π_χ (a Pontryagin space (^{8,9})). Relying on the theorem of M. G. Krein (^{10}) on expanding operators, we establish the following proposition:

Theorem 6. A simple dissipative K^r-operator A, acting in the space Π_χ, has a χ-dimensional negative invariant subspace.

We note that this assertion makes it possible to carry out the factorization of the characteristic matrix function and to construct a triangular model of a simple dissipative K^r-operator (by analogy with the way this was done in (^{2}) in the case of nonunitary operators).

In conclusion we note the following assertion (which can also be obtained without constructing the triangular model, on the basis of the preceding results and the results of (^{2})):

Theorem 7. Let A be a simple dissipative K^r-operator acting in the space Π_χ. Then

$$\det \chi_A(i) \leq \prod_{k=1}^{\chi} \left| \frac{\mu_k - i}{\mu_k + i} \right| \prod_{k=1}^{N} \left| \frac{\lambda_k - i}{\lambda_k + i} \right| \quad (N \leq \infty), \tag{4}$$

where $\{\mu_k\}_{k=1}^{\chi}$ and $\{\lambda_k\}_{k=1}^{N}$ are the totalities, respectively, of the (−)-proper and (+)-proper values of the operator A. Moreover, the system of finite-dimensional invariant subspaces of the operator A corresponding to the eigenvalues of this operator is complete in the space Π_χ if and only if equality holds in relation (4).

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Received
18 I 1966

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