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1968

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Abstract

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ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES OF ALMOST-PERIODIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov, May 19, 1967)

§ 1.

This note gives new sufficient criteria for the absolute convergence of Fourier series of numerical almost-periodic (a.p.) functions. These criteria, formulated in terms of best approximations, are adjacent to the theorems of papers ([5], [6]), which give sufficient conditions for the absolute convergence of Fourier series of a.p. functions in terms of moduli of continuity, under preliminary assumptions on the order of growth (decrease) of the Fourier exponents. The results presented contain generalizations of the theorems of Stechkin, Sas, and Bernstein ([4]) to a.p. functions. In § 3 the case of a single limit point of the spectrum of an a.p. function is considered; in § 4 the theorem of § 3 is extended to the case of any finite set of limit points of the spectrum.

§ 2.

Let us denote respectively by B^p, W^p, S_1^p $(p \geqslant 1)$, and U the classes of a.p. functions of Besicovitch, Weyl, Stepanov, and Bohr ([1], [2]). The inclusions

$$B \supset B^p \supset W^p \supset S_1^p \supset U,$$

hold, where B is the class B^1. Let $S(f) = \{\lambda_k\}$ be the set of Fourier exponents of a function $f(x) \in B$ (this set is called the spectrum of the function $f(x)$). In what follows it is assumed that $M\{f(x)\} = 0$ and that the limit points of the set $S(f)$ do not belong to the spectrum.

To a function $f(x) \in B$ with spectrum $S(f)$ there corresponds the Fourier series

$$f(x) \sim \sum_k A_{\lambda_k} e^{i\lambda_k x}. \tag{1}$$

For $f(x) \in B^2$ Parseval' s equality ([2]) holds:

$$M\{|f(x)|^2\} = \sum_k |A_{\lambda_k}|^2. \tag{2}$$

Let $f(x) \in B^2$; put

$$E_\lambda^{(2)}(f) = \inf_{g(x)\in P_\lambda^{(2)}} \{M[|f(x)-g(x)|^2]\}^{1/2},$$

$$\mathcal{E}_\lambda^{(2)}(f) = \inf_{g(x)\in Q_\lambda^{(2)}} \{M[|f(x)-g(x)|^2]\}^{1/2},$$

where $P_\lambda^{(2)} \subset B^2$ and $Q_\lambda^{(2)} \subset B^2$ are the classes of functions whose spectra belong respectively to the interval

$$I_\lambda = (-\lambda, \lambda)$$

and to the set

$$\tilde{I}_\lambda = (-\infty, -\lambda) \cup (\lambda, \infty) \quad (\lambda > 0).$$

On the basis of the Riesz-Fischer theorem ([2]) and by virtue of (1), (2),

$$E_\lambda^{(2)}(f) = \left\{ \sum_{|\lambda_k|\geqslant\lambda} |A_{\lambda_k}|^2 \right\}^{1/2}; \qquad \mathcal{E}_\lambda^{(2)}(f) = \left\{ \sum_{|\lambda_k|\leqslant\lambda} |A_{\lambda_k}|^2 \right\}^{1/2}; \tag{3}$$

as a consequence of (2),

$$\lim_{\lambda\to\infty} E_\lambda^{(2)}(f) = \lim_{\lambda\to 0} \mathcal{E}_\lambda^{(2)}(f) = 0.$$

Set

$$\omega^{(2)}(\delta, f) = \sup_{|h|\le\delta} \left[M_x\{|f(x+h)-f(x)|^2\}\right]^{1/2},$$

$$\tilde{\omega}^{(2)}(\delta, f) = \delta \left[M \left\{ \left| \int_0^\infty e^{-\delta t} f(x-t)\, dt \right|^2 \right\} \right]^{1/2}.$$

Lemma 1. For $f(x) \in B_2$

$$E_\lambda^{(2)}(f) \le \omega^{(2)}(2\pi/\lambda, f); \qquad \mathcal{E}_\lambda^{(2)}(f) \le \sqrt{2}\,\tilde{\omega}^{(2)}(\lambda, f). \tag{4}$$

Lemma 2. For $f(x) \in U$

$$E_\lambda^{(2)}(f) \le E_\lambda(f), \qquad \mathcal{E}_\lambda^{(2)}(f) \le \mathcal{E}_\lambda(f), \tag{5}$$

where

$$E_\lambda(f) = \inf_{g(x)\in P_\lambda} \left\{ \sup_x |f(x)-g(x)| \right\}, \qquad \mathcal{E}_\lambda(f) = \inf_{g(x)\in Q_\lambda} \left\{ \sup_x |f(x)-g(x)| \right\},$$

$P_\lambda \in U$ and $Q_\lambda \in U$ are classes of functions whose spectra belong respectively to the sets I_λ and $\tilde{I}_\lambda$.

§ 3. Let $S(f)$ have the unique limiting point $\Lambda^* = \infty$ or $\Lambda^* = 0$; by $\mathcal{L}(f) = \{\Lambda_k\}$ $(k = 1, 2, ...)$ we denote the monotone sequence of absolute values of the Fourier exponents of the function $f(x)$. In this case it is natural to write the Fourier series of $f(x) \in B$ in the symmetric form:

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\Lambda_k x} \quad (A_{-k} = -\Lambda_k;\ A_k = A_{\Lambda_k};\ A_k \cdot A_{-k} > 0 \text{ for } k \neq 0). \quad (1')$$

By virtue of (3), for $f(x) \in B^2$ having a Fourier series of the form $(1')$,

$$E_n^{(2)}(f) = \left\{ \sum_{|k| \geq n} |A_k|^2 \right\}^{1/2}, \qquad \mathcal{E}_{\Lambda_n}^{(2)}(f) = \left\{ \sum_{|k| \geq n} |A_k|^2 \right\}^{1/2}; \qquad (6)$$

the first of the equalities (6) holds for $\Lambda_k \uparrow \infty$, the second—for $\Lambda_k \downarrow 0$.

Theorem 1. Let $S(f)$ have the unique limiting point $\Lambda^* = \infty$ or $\Lambda^* = 0$.

The Fourier series $(1')$ of the function $f(x) \in B^2$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{E_{\Lambda_n}^{(2)}(f)}{\sqrt{n}} < \infty \qquad (\Lambda^* = \infty) \qquad (7)$$

or

$$\sum_{n=1}^{\infty} \frac{\mathcal{E}_{\Lambda_n}^{(2)}(f)}{\sqrt{n}} < \infty \qquad (\Lambda^* = 0). \qquad (7')$$

The proof of Theorem 1 is based on the application of the equalities (6) and Lemma 3 of paper ([3]) (see also ([2]) and the main theorem of paper ([12])).

In the case of a unique and finite limiting point of the spectrum $\Lambda^* \neq 0$, Theorem 1 is applicable to the function $f(x)e^{-i\Lambda^* x}$.

Corollary 1. Let $S(f)$ have the unique limiting point $\Lambda^* = \infty$. The Fourier series $(1')$ of the function $f(x) \in B^2$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{\omega^2(1/\Lambda_n, f)}{\sqrt{n}} < \infty. \qquad (8)$$

Proof. From (4) and (8) follows (7).

Corollary 1 contains Theorem 3 of paper ([5]) and the assertion of Theorem 3 of paper ([6]) on the absolute convergence of the Fourier series (1′) of a function $f(x) \in B^2$ under the conditions:

$$n = O(\Lambda_n), \qquad \sum_{n=1}^{\infty} \frac{\omega^{(2)}(1/n, f)}{\sqrt{n}} < \infty.$$

Corollary 2. Suppose that $S(f)$ has a unique limit point $\Lambda^* = \infty$. The Fourier series (1) of the function $f(x) \in U$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{E_{\Lambda_n}(f)}{\sqrt{n}} < \infty. \tag{9}$$

Proof. From (9) and (5) follows (7). (Estimates of $E_\lambda(f)$ analogous to Jackson' s theorems are contained in [7].)

§ 4. Suppose that $S(f)$ has a finite set of finite limit points Λ_j^* ($j = 1, 2, \dots, m;\ m \geqslant 1$). Fix $\eta > 0$ and, in the case of an unbounded spectrum, $M > 0$, satisfying the conditions: 1) the intervals $I_j = (\Lambda_j^* - 2\eta, \Lambda_j^* + 2\eta)$ ($j = 1, 2, \dots, m;\ m \geqslant 2$) do not intersect; 2) $I_j \subset [-M, M]$ ($j = 1, 2, \dots, m;\ m \geqslant 1$).

Renumber in decreasing order the points of the sets

$$\{|\lambda_k - \Lambda_j^*|\} \quad (\lambda_k \in S(f);\ 0 < |\lambda_k - \Lambda_j^*| \leqslant \eta;\ j = 1, 2, \dots, m);$$

as a result we obtain monotonically decreasing sequences

$$\mathcal{L}_j(f) = \{\Lambda_k^{(j)}\} \quad (k = 1, 2, \dots;\ \Lambda_k^{(j)} \downarrow 0;\ j = 1, 2, \dots, m).$$

If the spectrum is unbounded, renumber in increasing order the points of the set $\{|\lambda_k|\}$ ($\lambda_k \in S(f),\ |\lambda_k| \geqslant M$); as a result we obtain the sequence

$$\mathcal{L}_M(f) = \{\Lambda_k^{(M)}\} \quad (k = 1, 2, 3, \dots;\ \Lambda_k^{(M)} \uparrow \infty).$$

Lemma 3. If $f(x) \in B^2$ and has the Fourier series (1), then there exist functions $f_M(x) \in B^2$ and $f_{j,\eta}(x) \in B^2$ with Fourier series

$$f_M(x) \sim \sum_{|\lambda_k| \geq M} A_{\lambda_k} e^{i\lambda_k x}, \qquad f_{j,\eta}(x) \sim \sum_{|\lambda_k - \Lambda_j^*| \leq \eta} A_{\lambda_k} e^{i(\lambda_k - \Lambda_j^*)x} \quad (j = 1, 2, \dots, m). \tag{10}$$

If $f(x) \in U$, then $f_M(x) \in U,\ f_{j,\eta}(x) \in U$.

The first assertion of the lemma is a consequence of the Riesz-Fischer theorem [2], the second is contained in Lemma 2 of [10] (by virtue of Theorem 1 of note [11], the belonging of $f_{j,\eta}(x)$ to the class U follows from the condition $f(x) \in S_1^r$).

Lemma 4. Let $f(x) \in B^2,\ \varphi(x) \in B^2$, and for all real λ

$$|M\{\varphi(x)e^{-i\lambda x}\}| \leqslant |M\{f(x)e^{-i\lambda x}\}|;$$

then

$$E_\lambda^{(2)}(\varphi) \leqslant E_\lambda^{(2)}(f), \qquad \mathcal{E}_\lambda^{(2)}(\varphi) \leqslant \mathcal{E}_\lambda^{(2)}(f).$$

Theorem 2. Suppose that $S(f)$ has a finite set of finite limit points Λ_j^* $(j = 1, 2, \dots, m;\ m \geqslant 1)$. The Fourier series (1) of the function $f(x) \in B^2$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{E^{(2)}_{\Lambda_n^{(M)}}(f)}{\sqrt{n}} < \infty, \qquad \sum_{n=1}^{\infty} \frac{\mathcal{E}^{(2)}_{\Lambda_n^{(j)}}(f_j)}{\sqrt{n}} < \infty \quad (j = 1, 2, \dots, m), \tag{11}$$

where $f_j(x) = f(x)\{e^{-i\Lambda_j^* x}\}$ $(j = 1, 2, \dots, m)$.

Proof. By virtue of (10) and Lemma 4, $E^{(2)}_{\Lambda_n^{(M)}}(f_M) \ll E^{(2)}_{\Lambda_n^{(M)}}(f)$, $\mathcal{E}^{(2)}_{\Lambda_n^{(j)}}(f_{j,\eta}) \ll \mathcal{E}^{(2)}_{\Lambda_n^{(j)}}(f_j)$; therefore it follows from (11) that for $f_M(x)$ condition (7) is satisfied, and for $f_{j,\eta}(x)$ $(j = 1, 2, \dots, m)$ condition $(7')$ is satisfied. On the basis of Theorem 1,

$$\sum_{|\lambda_k| > M} |A_{\lambda_k}| < \infty, \qquad \sum_{|\lambda_k - \Lambda_j^*| \leq \eta} |A_{\lambda_k}| < \infty \quad (j = 1, 2, \dots, m).$$

Corollary 1. Let the spectrum $S(f)$ be bounded and have a finite number of limit points Λ_j^* $(j = 1, 2, \dots, m)$. The Fourier series (1) of the function $f(x) \in B^2$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{\tilde{\omega}^{(2)}(\Lambda_n^{(j)}, f_j)}{\sqrt{n}} < \infty \quad (j = 1, 2, \dots, m). \tag{12}$$

Proof. From (12) and (4) follows (11).

Corollary 1 of Theorem 2 contains Theorem 1 of the paper ([5]).

Corollary 2. Let the spectrum $S(f)$ be bounded and have a finite number of limit points Λ_j^* $(j = 1, 2, \dots, m)$. The Fourier series (1) of the function $f(x) \in U$ converges absolutely if

$$\sum_{n=1}^{\infty} \frac{\mathcal{E}_{\Lambda_n^{(j)}}(f_j)}{\sqrt{n}} < \infty \quad (j = 1, 2, \dots, m). \tag{13}$$

Proof. By virtue of (5) and (13), (11) holds. (Estimates of $\mathcal{E}_\lambda(f)$ are contained in the papers ([8,9]).)

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Received
11 V 1967

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Note: Figure translations are in progress. See original paper for figures.