SOME REMARKS ON LOGIC OF HIGHER ORDERS

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Abstract

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SOME REMARKS ON LOGIC OF HIGHER ORDERS

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In this note the results of ([1,2]) are strengthened.

We shall say that an ordinal α is **defined by a formula** σ (of finite or infinite order) if σ is true on all sets well ordered by type α, and only on them.

By induction we define the notion of a generally definable ordinal:

1. An ordinal definable by a formula of finite order is a generally definable ordinal.
2. If an ordinal α is defined by a formula such that the types of all variables occurring in it are generally definable ordinals, then α is a generally definable ordinal.

We shall call a formula **generally definable** if the types of all variables occurring in it are generally definable.

Theorem 1. *For every generally definable formula σ one can effectively construct a second-order formula $\Phi(\sigma)$ such that the truth (satisfiability, categoricity) of σ is equivalent to the truth (satisfiability, categoricity) of $\Phi(\sigma)$.*

This theorem strengthens Theorem 5 of ([2]) and, consequently, Hintikka' s result ([3]) on the reduction of type theory. From this theorem and a well-known result of A. Tarski ([4]) it follows that

Theorem 2. *For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas is not definable in arithmetic by any generally definable formula.*

Let, as in ([1,2]), $k(\sigma)$ be the least cardinal in the spectrum of the formula σ, and let k_0 be the upper bound of the set $\{k(\sigma)\}$, where $\{\sigma\}$ is the collection of all satisfiable second-order formulas. It is unknown whether k_0 coincides with the upper bound of the set $\{k(\tau)\}$, where $\{\tau\}$ is the set of all satisfiable formulas whose orders are less than k_0. In the case of coincidence, one can, assuming the axiom of choice, prove that for every ordinal $\alpha < k_0$ the set of numbers of true second-order formulas is not definable in arithmetic by any formula of order α.

At the same time, the set of numbers of true second-order formulas is definable in arithmetic by a formula of order $(k_0 + 1)$.

Let us express the result established by Theorem 2 in another form. We shall call a structure $\mathfrak{S}$ **generally definable** if there exists a generally definable formula true on all structures isomorphic to $\mathfrak{S}$, and only on them. We shall call a structure **arithmetized** if in it there is defined a sequence, of type ω, of elements $\Delta_0, \Delta_1, \ldots$. The set of natural numbers N will be called **definable** (generally definable) in an arithmetized structure $\mathfrak{S}$ if there exists a formula of finite order (a generally definable formula) σ in the signature $\mathfrak{S}$ such that $n \in N$ is equivalent to the truth of $\sigma(\Delta_n)$ for every natural n. For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas is not definable in any arithmetized structure definable by a second-order formula

order. If some set of natural numbers is generally definable in a generally definable arithmetized structure, then it is definable by a second-order formula in some structure definable by a second-order formula. Hence it follows that the set of numbers of true second-order formulas will not be generally definable in any generally definable arithmetized structure. Hence, and from the known theorem of Zykov ([5]), it follows:

Theorem 3. *For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas of the form*

$$(\exists X)(Y)\gamma,$$

where X is a binary and Y a unary predicate variable, and γ contains no quantifiers over predicate variables, is not generally definable in any generally definable arithmetized structure.

We shall call the **genus of a generally definable formula** the set of types of all variables occurring in it. We shall call the **genus of a system** Σ of generally definable formulas the union of the set of genera of the formulas from Σ.

The following theorem, proved with the aid of Theorem 1, generalizes Theorem 1 from ([2]).

Theorem 4. *Let Σ be a system of generally definable formulas with a finite set of nonlogical constants such that the set of degrees of formulas defining ordinals from the genus Σ is bounded by a generally definable ordinal, and the set of numbers of formulas from Σ (for some effective enumeration of the set of generally definable formulas) is defined in arithmetic by a generally definable formula. Then the system Σ is equivalent to some generally definable formula σ.*

Without giving the exact formulation, in view of its cumbersomeness, let us only note that the relation between the genus Σ and the genus σ is analogous to the relation between the degrees of a definable system of formulas and an equivalent formula, established by Theorem IV of ([1]).

We shall say that a set of ordinals $\mathfrak{D}$ is **defined by a formula** σ if σ is true on all sets from $\mathfrak{D}$ that are fully ordered by type, and only on them. We shall call a set of ordinals **generally definable** if it is defined by some generally definable formula. With the aid of Theorems 1 and 4 one proves:

Theorem 5. *For every generally definable formula σ and every generally definable set $\mathfrak{D}$ of generally definable ordinals, one can effectively construct second-order formulas $\Phi(\sigma)$ and $\Psi(\sigma)$ such that:*

1) *the equivalence of σ to some generally definable formula whose genus is included in $\mathfrak{D}$ is equivalent to the truth of $\Phi(\sigma)$;*
2) *the equivalence of σ to some system of generally definable formulas whose genus is included in $\mathfrak{D}$ is equivalent to the truth of $\Psi(\sigma)$.*

This theorem generalizes and strengthens Theorem VI of ([1]). It implies, in particular, that for every generally definable formula σ and every natural number n one can effectively construct second-order formulas $\Phi(\sigma)$ and $\Psi(\sigma)$ such that the equivalence of σ to some formula of the n-th order is equivalent to the truth of $\Phi(\sigma)$, and the equivalence of σ to some system of formulas of the n-th order is equivalent to the truth of $\Psi(\sigma)$.

The method of proof of Theorem 1 (see also the proof of Theorem 1.1 from ([1])) shows that the problem of characterizing the spectra of generally definable formulas reduces to the problem of characterizing the spectra of second-order formulas. The complexity of the latter problem is indicated, in part–

···that in the sentence of the strict axiom of infinity there occurs a formula of the 2nd order whose spectrum has the greatest cardinal, greater than k_0. With the aid of Theorems 1 and 4 one proves

Theorem 6. *For every recursively definable set $\mathfrak{D}$ of recursively definable ordinals there exists a formula of the 2nd order whose spectrum is not predicatively definable in the sense of* (2) *by any formula whose genus is included in $\mathfrak{D}$.*

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Note: Figure translations are in progress. See original paper for figures.