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Abstract

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MATHEMATICS

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ON MASLOV' S CANONICAL OPERATOR FOR HYPERBOLIC EQUATIONS

(Presented by Academician A. N. Tikhonov on 4 IV 1968)

Maslov' s canonical operator ([1])*, adapted to the study of solutions of equations of hyperbolic type, is obtained if, as the space H, one takes the space $L^2(E^1) \ni \varphi(\tau)$, and, as the operator A, the operator $-i\,d/d\tau$. In this case one may say that the space W_s, containing the range of the canonical operator K^α, consists of functions differentiable s times with respect to q and τ, defined up to functions having an indefinitely large number of derivatives with respect to q and τ. For applications to differential equations it is sufficient to know the result of the action of the canonical operator only for τ belonging to a neighborhood of zero. We shall introduce such realizations of the canonical operator whose values, for fixed q, differ by a function of τ holomorphic at $\tau = 0$. As an instrument we shall use the generalized method of stationary phase.

Let $f(x)$ be a function holomorphic at the point $x_0 \in E^n$; let $g(x)$ be a function with values in $L^2(E^1)$, holomorphic at the point x_0. Suppose that x_0 is an isolated stationary point of the real-valued function $f(x)$, and that, for any fixed x from some neighborhood of the point x_0, the function $g(x) = g(x;\tau)$ is holomorphic in τ on $(-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, together with all its derivatives.

Introduce the following notation. Let

$$F : L^2(E^1) \ni \varphi(\tau) \to F\varphi(\tau) = \tilde{\varphi}(\lambda) \in L^2(E^1)$$

be the Fourier transform. By e_ε we shall denote the operator in $L^2(E^1)$ which reduces to multiplication by the function $e_\varepsilon(\tau) = 1$ for $|\tau| < \varepsilon$ and 0 for $|\tau| > \varepsilon$, and by B_ε the operator in $L^2(E^1)$ acting according to the formula

$$B_\varepsilon\varphi(\tau) = ie_\varepsilon e^{\delta\tau} \int_0^\infty e^{-\delta y}\varphi(y)\,dy.$$

For brevity, instead of $R(A, -i\delta) = (A + i\delta)^{-1}$ we shall write simply R; δ will always be regarded as positive. The operator $[-i(A + i\delta)]^{-1/2}$ is by definition equal to

$$F^{-1}[-i(\lambda + i\delta)]^{-1/2}F,$$

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where $[-i(\lambda+i\delta)]^{1/2}$ denotes $e^{-i\pi/4}(\lambda+i\delta)^{1/2}$ for $\lambda>0$ and $e^{i\pi/4}(|\lambda|-i\delta)^{1/2}$ for $\lambda<0$; $(|\lambda|\pm i\delta)^{1/2}\to(|\lambda|)^{1/2}$ as $\delta\to+0$. Similarly,

$$[i(A-i\delta)]^{-1/2}\stackrel{\text{def}}{=}F^{-1}[i(\lambda-i\delta)]^{-1/2}F,$$

where $[i(\lambda-i\delta)]^{-1/2}$ denotes $e^{i\pi/4}(\lambda-i\delta)^{1/2}$ for $\lambda>0$ and $(-\lambda+i\delta)^{1/2}e^{-i\pi/4}$ for $\lambda<0$.

For the case $n=1$, in ([2]) the following is proved.

Theorem 1. Let $f''(x_0)>0$,

$$g_1(\xi)=g(x(\xi))e^{\delta f(x(\xi))}\frac{dx}{d\xi},\qquad g_{n+1}(\xi)=\frac{1}{2}\frac{d}{d\xi}\frac{g_n(\xi)-g_n(x_0)}{\xi-x_0},$$

$$\xi(x)=x_0+\big(f(x)-f(x_0)\big)^{1/2}.$$

There exist such positive numbers ε and μ that the equality is valid

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* All notation relating to the canonical operator completely coincides with the notation in ([1]).

$$e_\varepsilon e^{-i(A+i\delta)f(x_0)}\int_{x_0-\mu}^{x_0+\mu}e^{iAf(x)}g(x)\,dx=$$

$$=\sum_{n=1}^{\infty}\left\{\sqrt{\pi}\left(\frac{i}{2}\right)^{n-1}(e_\varepsilon R-B_\varepsilon)^{n-1}[-i(A+i\delta)]^{-1/2}g_n(x_0)\right\}+\chi,$$

where χ is a function holomorphic on $(-\varepsilon,\varepsilon)$, and the series converges uniformly in τ. Similarly one proves

Theorem 2. Let $f''(x_0)<0$,

$$g_1(\xi)=g(x(\xi))e^{-\delta f(x(\xi))}\frac{dx}{d\xi},\qquad g_{n+1}(\xi)=$$

$$=-\frac{1}{2}\frac{d}{d\xi}\frac{g_n'(\xi)-g_n'(x_0)}{\xi-x_0},\qquad \xi(x)=x_0+\big(f(x_0)-f(x)\big)^{1/2}.$$

There exist positive numbers ε and μ such that the equality

$$e_\varepsilon e^{i(-A+i\delta)f(x_0)}\int_{x_0-\mu}^{x_0+\mu}e^{iAf(x)}g(x)\,dx=$$

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$$= \sum_{n=1}^{\infty} \left\{ \sqrt{\pi} \left(\frac{i}{2}\right)^{n-1} (e_\varepsilon R - B_\varepsilon)^{n-1} [i(A - i\delta)]^{-1/2} g_n(x_0) \right\} + \chi,$$

where χ is a function holomorphic on $(-\varepsilon, \varepsilon)$, and the series converges uniformly in τ.

For the proof of this theorem the lemma from ([2]) must be modified as follows.

Lemma. Let $f(x)$ and $g(x)$ be holomorphic on $[x_0, b]$, $f'(x_0) = 0$, $f''(x_0) < 0$, $f'(x) \neq 0$ for $x_0 < x \leq b$; $g(x_0)$ and $g(b)$ are functions from $L^2(E^1)$, holomorphic on $(-\varepsilon - f(x_0) + f(b),\ \varepsilon - f(x_0) + f(b))$. Then

$$e^{-i(A-i\delta)f(x_0)} \int_{x_0}^{b} e^{iAf(x)} g(x)\, dx = \frac{\sqrt{\pi}}{2} [i(A - i\delta)]^{-1/2} g_1(x_0) +$$

$$+ \frac{i}{2} R \int_{x_0}^{b_1} e^{-i(A+i\delta)(\xi - x_0)^2} g_2(\xi)\, d\xi - \frac{i}{2} R g_1'(0) + \chi,$$

where $b_1 = x_0 + \left(f(x_0) - f(b)\right)^{1/2}$, χ is a function holomorphic on $(-\varepsilon, \varepsilon)$.

We pass to the case $n > 1$. Suppose that the matrix

$$R = \left\| \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right\|_{i,j=1}^{n}$$

is nonsingular and the quadratic form corresponding to this matrix is indefinite; the case of a sign-definite form $R_{ij} z^{iz^j}$ is reduced directly to Theorems 1 and 2. Let the matrix R have m positive and $n - m$ negative eigenvalues:

$$\lambda_1, \dots, \lambda_m > 0; \qquad \lambda_{m+1}, \dots, \lambda_n < 0.$$

In the monograph ([3]), on pp. 382-385, an analytic mapping $x \leftrightarrow u, v, \alpha, \beta;\ u \in E^1,\ v \in E^1,\ \alpha \in S^{m-1},\ \beta \in S^{n-m-1}$ is described such that

$$\int_{\omega} e^{iAf(x)} g(x)\, dx,$$

where ω is some neighborhood of the point x_0, is equal to

$$\frac{1}{|I|^{1/2}} e^{iAf(x_0)} \int d\Omega_\alpha d\Omega_\beta \int_0^a \int_0^b e^{iA(u^2 - v^2)} \tilde{g}(u, v, \alpha, \beta)\, du\, dv.$$

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Here $I = \det R$; $d\Omega_\alpha$ and $d\Omega_\beta$ are the area elements of the corresponding spheres. Denote by $G(u,v)$ the function $\int d\Omega_\alpha d\Omega_\beta\, \tilde{g}(u,v,\alpha,\beta)$ and introduce recursively the following countable family of functions:

$$G_1(u,v) = G(u,v)e^{\delta u^2}; \qquad G_n(u,u) = \frac{i}{2}\frac{\partial}{\partial u}\frac{G_{n-1}(u,v) - G_{n-1}(0,v)}{u};$$

$$G_{n,1}(u,v) = G_n(u,v)e^{\delta v^2}; \qquad G_{n,k}(u,v) = -\frac{1}{2}\frac{\partial}{\partial v}\frac{G_{n,k-1}(u,v) - G_{n,k-1}(u,0)}{v}.$$

Theorem 3. The following equality holds

$$e^{-iAf(x_0)}\int_\omega e^{iAf(x)}g(x)\,dx =$$

$$= \frac{1}{(|I|)^{1/2}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{\frac{\pi}{4}\left(\frac{i}{2}\right)^{n+k-2}(e_\varepsilon R - B_\varepsilon)^{n+k-2}[i(A-i\delta)]^{-1/2}\times\right.$$

$$\left.\times[-i(A+i\delta)]^{-1/2}G_{n,k}(0,0)\right\} + \chi,$$

where χ is a function holomorphic on $(-\varepsilon/2, \varepsilon/2)$.

Without describing all the details here, we give the main course of the proof. Choose $a \le (\varepsilon_0/2)^{1/2}$, $\varepsilon < a^2$. Applying the lemma from (2), we obtain

$$\int_0^a e^{iAu^2}G(u,v)\,du = \sum_{n=1}^{N}\frac{\sqrt{\pi}}{2}\left(\frac{i}{2}\right)^{n-1}(e_\varepsilon R - B_\varepsilon)^{n-1}[-i(A+i\delta)]^{-1/2}G_n(0,v)+$$

$$+\left(\frac{i}{2}\right)^{N}(e_\varepsilon R - B_\varepsilon)^{N}\int_0^a e^{i(A+i\delta)u^2}G_{N+1}(u,v)\,du + \chi_N(v).$$

Integrating with respect to v, we get

$$\int_0^a\int_0^b e^{iA(u^2-v^2)}G(u,v)\,dv =$$

$$= \sum_{n=1}^{N}\frac{\sqrt{\pi}}{2}\left(\frac{i}{2}\right)^{n-1}\int_0^b e^{-iAv^2}(e_\varepsilon R - B_\varepsilon)^{n-1}[-i(A+i\delta)]^{-1/2}G_n(0,v)\,dv+$$

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$$+\left(\frac{i}{2}\right)^N \int_0^b dv\, e^{-iAv^2}(e_\varepsilon R - B_\varepsilon)^N \int_0^a e^{i(A+i\delta)u^2} G_{N+1}(u,v)\, du + \int_0^b e^{-iAv^2}\chi_N(v)\, dv. \tag{1}$$

It can be shown that the last integral is holomorphic on $(-\varepsilon, \varepsilon)$. Applying the lemma to the sum

$$\sum_{n=1}^{N} \frac{\sqrt{\pi}}{2}\left(\frac{i}{2}\right)^{n-1} \int_0^b e^{-iAv^2}(e_\varepsilon R - B_\varepsilon)^{n-1}[-i(A+i\delta)]^{-1/2} G_n(0,v)\, dv,$$

we transform it into the form

$$\sum_{n=1}^{N}\sum_{k=1}^{K}\left\{\frac{\pi}{4}\left(\frac{i}{2}\right)^{n+k-2}(e_\varepsilon R - B_\varepsilon)^{n+k-2}[i(A-i\delta)]^{-1/2}[-i(A+i\delta)]^{-1/2} G_{n,k}(0,0)\right\} +$$

$$+\sum_{n=1}^{N}\left\{\frac{\sqrt{\pi}}{2}\left(\frac{i}{2}\right)^{n}(e_\varepsilon R - B_\varepsilon)^{n-1}[-i(A+i\delta)]^{-1/2}\left(\frac{i}{2}\right)^{K}(e_\varepsilon R - B_\varepsilon)^{K}\times\right.$$

$$\left.\times\int_0^b e^{-i(A-i\delta)v^2} G_{n,K+1}(0,v)\, dv\right\} + \chi, \tag{2}$$

where χ is holomorphic on $(-\varepsilon/2, \varepsilon/2)$. We have used the fact that the commutator of the operators $e_\varepsilon R - B_\varepsilon$ and $[i(A-i\delta)]^{-1/2}$ maps any function from $L^2(E^1)$ into a function holomorphic on $(-\varepsilon/2, \varepsilon/2)$. The last sum in (2), for fixed N, can be made arbitrarily small by choosing K. The term

$$\left(\frac{i}{2}\right)^N \int_0^b dv\, e^{-iAv^2}(e_\varepsilon R - B_\varepsilon)^N \int_0^a e^{i(A+i\delta)u^2} G_{N+1}(u,v)\, du$$

in (1) can be made arbitrarily small by choosing sufficiently large-

some N. The proof of the uniform, in τ on $(-\varepsilon, \varepsilon)$, convergence of the series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\pi}{4}\left(\frac{i}{2}\right)^{n+k-2}(e_\varepsilon R - B_\varepsilon)^{n+k-2}[i(A+i\delta)]^{-1/2}[-i(A+i\delta)]^{-1/2} G_{n,k}(0,0)$$

and the estimates of the terms in (1) and (2) are carried out analogously to the way this was done in ([2]), namely: the estimates of the norms of the functions G_n

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and $G_{n,k}$ are obtained by induction, and then the inequality $\|(e_\varepsilon R - B_\varepsilon)^n \varphi(\tau)\| \leq e^{n\delta\varepsilon}\sqrt{\varepsilon}\tau^{n-1}[(n-1)!]^{-1}\|\varphi\|$, valid for every $\varphi \in L^2(E^1)$, is used. From the proved theorems on the method of stationary phase there follows the following assertion on the canonical operator for hyperbolic equations. We shall call a special realization of the operator $V_{i_1\ldots i_k}^{i_m\ldots i_k}$ introduced in (1) an operator having the property that, after multiplying it on the left by e_ε, one obtains the sum of the series formed from the formal series which defines the operator $V_{i_1\ldots i_k}^{i_m\ldots i_k}$ in (1), by replacing R by $e_\varepsilon R - B_\varepsilon$. We note that the series thus modified is convergent. To a special realization of the operator $V_{i_1\ldots i_k}^{i_m\ldots i_k}$ there corresponds a special realization of the canonical operator.

Theorem 4. Let the point q_0 belong to the image of the element $u_j \subset M^n$ under the projection of M^n onto E^n; let all elements of the weight decomposition of unity be holomorphic on u_j, and let the representatives $\widetilde{\varphi}_{u_j}$ and $\widetilde{\widetilde{\varphi}}_{u_j}$ of the function φ_{u_j} be holomorphic and differ from one another by a holomorphic function in a neighborhood of the preimage of the point q_0, and, moreover, let they themselves and their derivatives for fixed argument be functions of τ, holomorphic in some punctured neighborhood of zero, possibly with the exception of zero itself. Then, for two special realizations $\widetilde{K}^{\alpha^0}$ and $\widetilde{\widetilde{K}}^{\alpha^0}$ of the canonical operator, the function $(\widetilde{K}^{\alpha^0}\varphi_{u_j})(q_0) - (\widetilde{\widetilde{K}}^{\alpha^0}\widetilde{\widetilde{\varphi}}_{u_j})(q_0)$ is holomorphic at the point 0.

Remark. The choice of sufficiently small ε in the special realizations of the canonical operator is understood.

Thus, one may say that the canonical operator is defined up to functions holomorphic in τ in a neighborhood of the point 0.

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[1] V. P. Maslov, DAN, **177**, No. 6, 1277 (1967). [2] V. L. Dubnov, UMN, **23**, issue 1 (139), 223 (1968). [3] V. P. Maslov, *Theory of Perturbations and Asymptotic Methods*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.

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