THE STRUCTURE OF THE BASIC CLASSES OF SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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Abstract

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Z. I. KOZLOVA

THE STRUCTURE OF THE BASIC CLASSES OF SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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In the spaces D^τ, J^τ and in the topological τ-spaces $D^{\omega_\nu}, J^{\omega_\nu}$ we have constructed classes of B-, A-, C-, R-, P-sets and have studied their simplest properties ([1], [2]). We shall study the general properties of these classes of sets.

Let $I = (i)$ be a set, called the space of indices, of cardinality $\tau = \aleph_\nu$. We regard the number τ as a strongly inaccessible cardinal number.

A set E of a topological space $\mathfrak{N}_{xy}$ is called universal for the class $K \subset \mathfrak{P}\mathfrak{N}_y$ if: 1) it is homeomorphic to one of the sets of the class K; 2) for whatever set $M \in K$, there exists a point $x_0 \in \mathfrak{N}_x$ such that $E \cap (x_0 \times \mathfrak{N}_y)$ is congruent to M.

If E is a universal set for the class K, then CE is a universal set for the class K^c.

Theorem 1. *If the topological space $\mathfrak{N}_x$ of weight τ is homeomorphic to the space $\mathfrak{N}_{x_0 \dots x_i \dots y}$, then the classes F and G of this space have a universal set.*

Corollary 1. *Each of the classes F and G of the spaces $D^\tau, J^\tau, D^{\omega_\nu}, J^{\omega_\nu}$ has a universal set.*

A family of sets $(E_i)_i$ of a class K of a topological space $\mathfrak{N}_{xy}$ is called a universal family of sets for this class of the space $\mathfrak{N}_y$, if for every family of sets $(\mathcal{E}_i)_i$ of the class $K \subset \mathfrak{P}\mathfrak{N}_y$ there is a point $x_0 \in \mathfrak{N}_x$ such that

$$\mathrm{pr}_y(E_i \cap (x_0 \times \mathfrak{N}_y)) = \mathcal{E}_i$$

for $i \in I$.

If $(E_i)_i$ is a universal family of sets, then every subfamily of it and every set of this family (the family of complementary sets) is universal for the class K (K^c). From the investigations of L. V. Kantorovich and E. M. Livenson ([4]) it follows that if a topological space $\mathfrak{N}_x$ of weight τ is homeomorphic to the topological space $\mathfrak{N}_{x_0 \dots x_i \dots}$, and the class of sets $K \subset \mathfrak{P}\mathfrak{N}$ has a universal set and is invariant with respect to homeomorphic transformations, then the class K also has a universal family of sets. Hence we obtain

Corollary 2. *For each of the classes F and G of the spaces under study there exists a universal family of sets.*

L. V. Kantorovich and E. M. Livenson also proved that if the class of sets $K \subset \mathfrak{P}\mathfrak{N}_{xy}$ has a universal family of sets and if Ψ_N is a certain set-theoretic operation, then the class $\Psi_N(K)$ also has a universal family of sets. Hence we obtain:

Corollary 3. *Each of the following classes of sets of the spaces under study has a universal family of sets: 1) F^α and G^α for $\alpha < \omega_{\nu+1}$; 2) R_α and CR_α for $\alpha < \omega_{\nu+1}$; 3) $R_{\beta\alpha}$ and $CR_{\beta\alpha}$ for $\alpha, \beta < \omega_{\nu+1}$, in particular, C_α and $C(C_\alpha)$ for $\alpha < \omega_{\nu+1}$; 4) $\widetilde{R}_\alpha$ and $C\widetilde{R}_\alpha$ for $\omega_{\nu+1} \leq \alpha < \omega_{\nu+2}$; 5) P_α and CP_α for $\alpha < \omega_{\nu+1}$.*

From G. Cantor' s diagonal lemma it follows that: 1) for the class of B-sets of the spaces under study there does not exist a universal set; 2) if a class of sets K of a topological space $\mathfrak{N}$ has a universal set E, then $CE \notin K$. Hence we obtain:

Corollary 4. In each class of sets $F^\alpha, G^\alpha, CR_\alpha, R_\alpha, R_{\alpha\beta}, CR_{\alpha\beta}, P_\alpha, CP_\alpha$ for $\alpha, \beta < \omega_{\nu+1}$, $\bar{R}_\alpha, C\bar{R}_\alpha$ for $\omega_{\nu+1} \leqslant \alpha < \omega_{\nu+2}$, there exists a set different from the sets of the preceding classes.

The question of the essential extension of the classes of sets under study can also be settled on the basis of the following theorem:

Theorem 2. If a topological space $\mathfrak{R}$ contains a subset homeomorphic to the space $D^\tau(D^{\omega_\nu})$, K is the class of all closed subsets of the space $\mathfrak{R}$, and Ψ_N is an arbitrary set-theoretic operation, then the class of sets $\Psi_N(K)$ is not invariant with respect to the operation of complementation.

Let us note that the space $J^{\omega_\nu}(J^\tau)$ contains topologically $D^{\omega_\nu}(D^\tau)$.

Corollary 5. There is no $\Delta\Sigma$-operation Φ_N yielding all and only the B-sets in the spaces under study.

It was proved by V. P. Filippov ([5]) that a homeomorphism between sets E_1 and E_2 in complete regular uniform spaces with a fundamental system of entourages of cardinality τ can be extended to a homeomorphism between sets of type G_Δ. For $\tau = \aleph_0$ we have the theorem of M. A. Lavrent' ev ([6]). Hausdorff' s theorem on topological invariance ([7]) is also generalized to these spaces.

Theorem 3. If F is the class of closed sets of a complete regular uniform space $\mathfrak{R}$ with a fundamental system of entourages of cardinality τ, and Ψ_N is a pseudo-disjunction ([8]) such that the intersection of any set from $\Psi_N(F)$ with any set of type G_Δ itself belongs to $\Psi_N(F)$, then the class of sets $\Psi_N(F)$ is invariant with respect to homeomorphic transformation.

Corollary 6. The following classes of sets of the spaces under study are topologically invariant:
1) F^α for $\alpha \geqslant 2$;
2) G^α for $\alpha \geqslant 3$;
3) $(\omega_\nu)A$ and $(\omega_\nu)CA$;
4) $(\omega_\nu)R_\alpha$, $(\omega_\nu)CR_\alpha$ for $\alpha < \omega_{\nu+1}$ and their subclasses, in particular the classes $(\omega_\nu)C_\alpha$ and $(\omega_\nu)C(C_\alpha)$;
5) $\bar{R}_\alpha$ and $C\bar{R}_\alpha$ for $\omega_{\nu+1} \leqslant \alpha < \omega_{\nu+2}$;
6) $(\omega_\nu)P_\alpha$ and $(\omega_\nu)CP_\alpha$ for $\alpha < \omega_{\nu+1}$.

V. P. Filippov also investigated the question of the existence of canonical elements in each class of B-sets of the spaces J^{ω_ν} and D^{ω_ν}.

Let us consider the question of B-approximability of sets of various classes of the basic set-theoretic hierarchies in the space J^{ω_ν}. A portion of a set E is any nonempty intersection of it with some Baire interval of this space.

Theorem 4. In order that a set $E \in B^2 \subset \mathfrak{P}J^{\omega_\nu}$, it is necessary and sufficient that on every perfect set P of this space there exist a portion wholly belonging either to E or to CE.

A set $E \subset J^{\omega_\nu}$ has the Baire property if on every perfect set P there exists a portion on which either the set E itself or its complement CE is of the first category. One may also say that a set $E \subset J^{\omega_\nu}$ has the Baire property if, whatever the perfect set P is, either E is of the first category on P, or there exists a portion of P on which CE is of the first category. Let us note that the class θ of all sets of the first category on some perfect set P is a B-hereditary $\underset{\tau}{\cup}$-class, and $\theta \not\ni P$.

Theorem 5. Every B-set of the space J^{ω_ν} has the Baire property.

Theorem 6. Whatever the perfect set $P \subset J^{\omega_\nu}$ and the system $\mathfrak{M}$ of pairwise disjoint subsets of it having the Baire property, the aggregate of those subsets which are not of the first category on P has cardinality $\leqslant \tau$.

Theorem 7. Every set having the Baire property differs on every perfect set P from some G_Δ only by a set of the first category on P.

Corollary 7. The class θ of all sets of the first category in some perfect set P is a B-resolved class; moreover, the (B, θ)-approximable sets are precisely all sets possessing the Baire property.

Corollary 8. The class of B-sets of the space J^{ω_ν} is a class of B-approximable sets.

Corollary 9. The classes of $(\omega)A$-, $(\omega)CA$-, $(\omega)R$-, $(\omega)CR$-, $(\omega)C$-, $(\omega)\bar{R}$-, $(\omega)C\bar{R}$-sets of the space J^{ω_ν} are classes of B-approximable sets.

Corollary 10. The decomposition of $(\omega)A$-, $(\omega)CA$-, $(\omega)R$-, $(\omega)CR$-sets into internal and external constituents is regular with respect to the B-resolved class θ.

The theorems on the B-approximability of sets also hold for the spaces D^{ω_ν}, D^τ, J^τ.

A set $E \subset J^{\omega_\nu}$ will be called a set of full cardinality if $\mathrm{Card}(E) > \tau$. Every perfect set of the space J^{ω_ν} has cardinality 2^τ.

Theorem 8. Every $(\omega_\nu)A$-set of full cardinality in the space J^{ω_ν} has cardinality 2^τ.

Corollary 11. Every B-set of full cardinality in the space J^{ω_ν} contains a perfect subset and has cardinality 2^τ.

Theorem 8 and its corollary also hold for the spaces D^{ω_ν}, D^τ, J^τ.

Volgograd Pedagogical Institute
named after A. S. Serafimovich

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Note: Figure translations are in progress. See original paper for figures.