# THE CAUCHY PROBLEM FOR OPERATOR-DIFFERENTIAL EQUATIONS

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## Abstract

## Full Text

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**MATHEMATICS**

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# THE CAUCHY PROBLEM FOR OPERATOR-DIFFERENTIAL EQUATIONS

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The paper studies the abstract Cauchy problem for equation (1) in a Hilbert space $H$. Clearly, for $s > 2$ this problem is, as a rule, ill-posed in the sense of Hadamard–Petrovskii. In § 1 it is established that the Cauchy problem for equation (1) is well-posed in the sense of A. N. Tikhonov (see, for example, ([1],[2])). Next, a regularizer for the Cauchy problem is constructed (the Cauchy problem with additional unknown functions). In § 2 well-posed problems (called boundary-value problems) are considered which do not require additional unknowns.

It should be said that the abstract Cauchy problem for equations of the form (1) with $s > 2$ was first studied by E. Hille (see, for example, ([3])). In E. Hille' s works it was noted that the Cauchy problem (1), (2) is solvable only under a finite number of additional operator relations, called the defect of the problem. From the results of the present paper it follows, in particular, that the defect of the abstract Cauchy problem is equal to the number $k$ from Definition 2.

Finally, we note that in studying equation (1) it proved convenient to divide it into two classes: equations of parabolic type and equations of hyperbolic type.

§ 1. Let $u(t) : [0, \infty) \to H$,

$$P_s\left(\frac{d}{dt}\right) u \equiv \sum_{q=0}^{s} a_q u^{(q)} *,$$

where $a_q \in \mathbf{C}^1$, $s \geq 1$, $a_s = \pm 1$, $u^{(q)} = d^q u/dt^q$; $A(u) : H \to H$ is a closed operator.

Consider the Cauchy problem

$$\mathfrak{A}(u) \equiv P_s\left(\frac{d}{dt}\right) u + A(u) = h(t), \tag{1}$$

$$u(0) = 0, \dots, u^{(s-1)}(0) = 0. \tag{2}$$

Notation:

$$H^l_\gamma(A) = \left\{ u(t) \;\middle|\; \|u\|^2_{l,\gamma}(A) = \int_0^\infty \left(\|u^{(l)}\|^2 + \|A(u)\|^2\right) e^{-\gamma t}\, dt < \infty \right\};$$

$$\overset{\circ}{H}{}^l_\gamma(A) = \{u(t) \mid u(t) \in H^l_\gamma(A), \quad u(0) = 0, \dots, u^{(l-1)}(0) = 0\};$$

$$H^l_\gamma(\overset{\circ}{H}{}^l_\gamma) \equiv H^l_\gamma(A)(\overset{\circ}{H}{}^l_\gamma(A)) \quad \text{when } A \equiv 0.$$

Here $\|u\|$ is the norm of $u(t)$ in $H$.

Assume now that $A(u)$ is a self-adjoint operator with purely point spectrum $\lambda_0, \lambda_1, \dots$ of finite multiplicity, $\lambda_0 \le \lambda_1 \le \cdots$, whose eigenfunctions form a basis in $H$.

**Definition 1.** The operator $\mathfrak{A}(u)$ is called **strictly parabolic** for $\gamma \in \Gamma_0 \subset \mathbf{R}^1$ if for every $\gamma \in \Gamma_0$

$$A(\gamma; \tau) \equiv \operatorname{Re} \sum_{0 \le p \le q = 0}^{s} a_q \beta_{pq} (-i\tau)^p + \lambda_0 > 0 **,$$

where

$$\beta_{pq} = \frac{q!}{(q-p)!\, p!} \left(\frac{\gamma}{2}\right)^{q-p}.$$

* Differentiation is understood in the sense of the theory of generalized functions ([4]).

** It is easy to see that $A(\gamma; \tau) = \operatorname{Re} P_s(i\tau + \gamma/2) + \lambda_0$.

**Theorem 1.** If the operator $\mathfrak{A}(u)$ is strictly parabolic, then for $\gamma \in \Gamma_0$ the Cauchy problem (1), (2) is well posed in the sense of Tikhonov; moreover, for $h(t) \in H^l_\gamma$ the solution $u(t) \in H^{l+s}_\gamma(A)$, and the inequality

$$\|u\|_{l+s,\gamma}(A) \le C \|h\|_{l,\gamma}$$

holds.

**Remark.** For this theorem the preceding assumptions concerning the operator $A(u)$ are immaterial; in Definition 1 one need only set

$$\lambda_0 = \inf_{u \in H} \frac{(A(u), u)}{\|u\|^2}.$$

**Definition 2.** The **generalized Cauchy problem** for equation (1) is the problem of finding, for a given function $h(t) \in H^k_\gamma$, a system of functions $u(t) \in H^s_\gamma(A) \cap H^{s+k}_\gamma(A)$, $\rho_i \in H,\ i = 1, \dots, k$, such that

$$P_s\left(\frac{d}{dt}\right)u + A(u) = h(t) + P_{k-1}(t),$$

where $P_{k-1}(t) = \rho_1 + \rho_2 t + \dots + \rho^{k-1}_{kt}$. Here $k = [s/2]$, if $s$ is even; $k = [s/2]$, if $s$ is odd and $(-1)^{[s/2]}a_s > 0$; $k = [s/2] + 1$, if $s$ is odd and $(-1)^{[s/2]}a_s < 0$.

**Theorem 2.** For every $h(t) \in H^k_\gamma$ there exists a unique solution of the generalized Cauchy problem, and

$$\|u^{(k)}\|_{s,\gamma}(A) + \sum_{i=1}^{k} \|\rho_i\| \le C\|h\|_{k,\gamma}.$$

**Theorem 3.** The generalized Cauchy problem is a regularizer for the Cauchy problem (1), (2). Moreover, if $u_\varepsilon(t)$ is the solution of the generalized mixed problem for $h(t) = h_\varepsilon(t)$, then

$$\|u^{(k)} - u^{(k)}_\varepsilon\|_{s,\gamma}(A) \le C\|h^{(k)} - h^{(k)}_\varepsilon\|_{0,\gamma}.$$

§ 2. Closely connected with the generalized Cauchy problem is the following boundary-value problem: for a given function $h(t) \in H^0_\gamma$, find a solution of equation (1) $u(t) \in H^s_\gamma(A)$ such that

$$u(0) = 0, \dots, u^{(s-1-k)}(0) = 0.$$

**Theorem 4.** If $\gamma \in \Gamma_0$, then the boundary-value problem is uniquely solvable, and

$$\|u\|_{s,\gamma}(A) \le C\|h\|_{0,\gamma}.$$

Let us now consider the preceding questions for arbitrary $\gamma$.

**Definition 3.** The operator $\mathfrak{A}(u)$ is called **parabolic on the set** $\gamma \in \Gamma \subset R^1$ if for every $\gamma \in \Gamma$ there exists a number $\alpha(\gamma) \ge 0$ such that

$$\mathrm{Re} \sum_{0 \le p \le q = 0}^{s} a_q \beta_{pq} (-i\tau)^p + \alpha(\gamma) > 0.$$

**Theorem 5.** If the operator $\mathfrak{A}(u)$ is parabolic on the set $\Gamma$, then for every $\gamma \in \Gamma$ the boundary-value problem is normally solvable. This means that the problem is solvable if a finite number of additional conditions are imposed on the right-hand side $h(t)$; moreover, the kernel of the problem may be a finite-dimensional subspace.

To prove this theorem, let us note that if by $H_\gamma^+$ we denote the subspace $H_\gamma$ generated by the eigenfunctions $\omega_i$ of the operator $A(u)$ for which $\lambda_i > \alpha(\gamma)$, then the restriction of the boundary-value problem to this subspace, by virtue of Theorem 4, determines an isomorphism. Since the orthogonal complement to $H_\gamma^+$ is finite-dimensional, the kernel and cokernel of the original boundary-value problem are finite-dimensional.

§ 3. Let us now consider hyperbolic equations.

**Definition 4.** The operator $\mathfrak{A}(u)$ is called **strictly hyperbolic on the set** $\Gamma_0 \subset \mathbf{R}^1$, if for every $\gamma \in \Gamma_0$

$$A_1(\gamma;\tau) \equiv \operatorname{Re} \sum_{\substack{0\le p\le q=0\\ 0\le l\le r\le 1}}^{s} a_q \beta_{pq} \beta_{lr} (-i\tau)^{p+l} + \frac{\gamma}{2}\lambda_0 > 0^*.$$

**Definition 5.** The operator $\mathfrak{A}(u)$ is called **hyperbolic on the set** $\Gamma \subseteq \mathbf{R}^1$, if for every $\gamma \in \Gamma$ there exists a constant $\alpha(\gamma) \ge 0$ such that

$$\operatorname{Re} \sum_{\substack{0\le p\le q=0\\ 0\le l\le r\le 1}}^{s} a_q \beta_{pq} \beta_{lr} (-i\tau)^{p+l} + \alpha(\gamma) > 0.$$

For hyperbolic equations one can construct the entire theory of the preceding paragraphs; we note only that in all estimates on the right-hand side smoothness in $t$ greater by one unit is required.

**Remark.** Definitions 1 and 4 depend substantially on the lower-order terms of the operator $\mathfrak{A}(u)$; on the contrary, in Definitions 3 and 5 only the highest derivatives play a role. Elementary considerations lead to the fact that the operator $\mathfrak{A}(u)$ is parabolic for odd $s$, and for even $s$ when $(-1)^{[s/2]} a_s > 0$. In the case of even $s$ and $(-1)^{[s/2]} a_s < 0$, the operator $\mathfrak{A}(u)$ is hyperbolic.

§ 4. **Examples.** Let

$$A(u) \equiv A(x,D)u = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha u$$

be an elliptic differential operator of order $2m$, considered in a domain $G \subset \mathbf{R}^n$. On the boundary $\partial G$ there are self-adjoint boundary conditions

$$B_j(x,D)u\big|_{\partial G} = 0, \qquad j = 1,\dots,m$$

(the order of the boundary operators $B_j(x, D)u$ is not greater than $2m - 1$).

1. Let

$$\partial u/\partial t + A(x, D)u = h(x, t).' \tag{3}$$

In this case the boundary-value problem is the mixed problem for equation (3) in the cylinder $Q = G \times [0, \infty)$, $S = \partial G \times [0, \infty)$:

$$u(x, 0) = 0, \qquad B_j(x, D)u\big|_S = 0, \qquad j = 1, \dots, m.$$

We have $A(\gamma; \tau) \equiv \gamma/2 + \lambda_0$. Hence it follows that for $\gamma > -2\lambda_0$ the mixed problem is uniquely solvable, while for $\gamma \leq -2\lambda_0$ it has normal solvability.

2. Let

$$-\partial u/\partial t + A(x, D)u = h(x, t). \tag{4}$$

In this case the boundary-value problem consists in finding, in the space $H^s_\gamma(A)$, a solution of equation (4) satisfying on $S$ the boundary conditions

$$B_j(x, D)u\big|_S = 0, \qquad j = 1, \dots, m,$$

and free from any conditions at $t = 0$. We have $A(\gamma; \tau) \equiv -\gamma/2 + \lambda_0$. Consequently, $\Gamma_0 = (-\infty, 2\lambda_0)$. Outside $\Gamma_0$ the problem is normally solvable.

3. Let

$$-\partial^2 u/\partial t^2 + A(x, D)u = h(x, t). \tag{5}$$

* It is easy to see that $\quad A_1(\gamma; \tau) \equiv \mathrm{Re}(-i\tau + \gamma/2)P_s(i\tau + \gamma/2) + \lambda\gamma/2.$

Then $A(\gamma; \tau) \equiv \tau^2 - \gamma^2/4 + \lambda_0$. Hence, for $\lambda_0 > 0$, the problem

$$u(x, 0) = 0, \qquad B_j(x, D)u\big|_S = 0, \qquad j = 1, \dots, m \tag{6}$$

is uniquely solvable for $|\gamma| < 2\sqrt{\lambda_0}$. For $|\gamma| \geq 2\sqrt{\lambda_0}$ in the case $\lambda_0 > 0$, and for all $\gamma$ in the case $\lambda_0 \leq 0$, problem (5), (6) is normally solvable.

4. Let

$$\partial^2 u/\partial t^2 + A(x,D)u = h(x,t).$$

Then

$$A_1(\gamma;\tau) \equiv \frac{\gamma}{2}\left(\tau^2 + \frac{\gamma^2}{4} + \lambda_0\right).$$

It follows that the problem

$$u(x,0) = 0, \qquad u'(x,0) = 0, \qquad B_j(x,D)u\big|_S = 0, \qquad j = 1,\dots,m,$$

is uniquely solvable if $\gamma^2 > -4\lambda_0$ and $\gamma > 0$. In the case $\lambda_0 < 0$, for $\gamma \in (0, 2\sqrt{-\lambda_0})$ there is normal solvability.

5. Let

$$(-1)^k u^{(2k+1)} + A(x,D)u = h(x,t), \tag{7}$$

$$u(x,0) = 0,\dots,u^{(k)}(x,0) = 0, \qquad B_j(x,D)u\big|_S = 0. \tag{8}$$

We have

$$A(\gamma;\tau) \equiv \frac{\gamma}{2}\tau^{2k} + a_1(\gamma)\tau^{2k-2} + \dots + \lambda_0.$$

Consequently, for $\gamma \in (0,\gamma_0)$, where $\gamma_0$ is the lower bound of the $\gamma$ for which $A(\gamma;\tau) > 0$, problem (7), (8) is uniquely solvable. For any $\gamma \geq \gamma_0$ there is normal solvability.

6. Let

$$(-1)^{k-1}(u^{(2k+1)} - au^{(2k)}) + A(x,D)u = h(x,t), \tag{9}$$

$$u(x,0) = 0,\dots,u^{(k-1)}(x,0) = 0, \qquad B_j(x,D)u\big|_S = 0. \tag{10}$$

We have

$$A(\gamma;\tau) \equiv (-\gamma/2 + a)\tau^{2k} + a_1(\gamma)\tau^{2k-2} + \dots + \lambda_0.$$

Thus, for $\gamma < \gamma_0$, problem (9), (10) is uniquely solvable, where $\gamma_0 < 2a$ is the upper bound of the $\gamma$ for which $A(\gamma;\tau) > 0$. On the other hand, for any $\gamma < 2a$ there is normal solvability of problem (9), (10).

7. Let

$$(-1)^k u^{(2k)} + A(x, D)u = h(x, t). \tag{11}$$

We have

$$A(\gamma;\tau) \equiv \tau^{2k} + a_1(\gamma)\tau^{2k-2} + \cdots + \lambda_0.$$

Consequently, problem (11), (10) is normally solvable for all $\gamma$. For those $\gamma$ for which $A(\gamma;\tau) > 0$, there is unique solvability.

8. Let

$$(-1)^{k-1} u^{(2k)} + A(x, D)u = h(x, t). \tag{12}$$

We have

$$A_1(\gamma;\tau) \equiv \frac{\gamma}{2}\left(t^{2k} + a_1(\gamma)t^{2k-2} + \cdots + \lambda_0\right).$$

Consequently, problem (12), (8) is normally solvable for any $\gamma > 0$ and is uniquely solvable for those $\gamma > 0$ for which $A_1(\gamma;\tau) > 0$.

*Note added in proof.* At present the results of this work have been completely extended to the case of a non-self-adjoint operator $A$ in the space $L_p$.

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*Note: Figure translations are in progress. See original paper for figures.*