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Abstract

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ON ONE METHOD FOR SOLVING A STATIONARY PROBLEM

For the solution of stationary problems, an effective apparatus of one-step and multistep iterative methods has been developed ([1]). Of interest are multistep processes for which the functional of the solution tends uniformly to the solution of stationary problems. Various iterative processes of this type are known, based on the averaging method ([2–4]).

In the present note a two-step iterative process with a special averaging operator is considered; it turns out to be very effective for the numerical solution of problems of mathematical physics in cases where the ratio of the maximal eigenvalue of a positive matrix to the minimal one is very large.

1°. Suppose we have the stationary problem

$$\mathcal{L}\varphi = f, \tag{1}$$

$$\varphi|_s = 0, \tag{2}$$

where $\mathcal{L}$ is a linear nonsingular operator with domain of definition $G(\mathcal{L}) \in \mathcal{L}_2(D)$, D is a finite simply connected domain in an n-dimensional Euclidean space with boundary s.

To solve this problem we shall consider a nonstationary problem of the form

$$\partial^2\psi/\partial t^2 + \mathcal{L}\psi = f, \tag{3}$$

$$\psi|_{t=0} = \psi'_t|_{t=0} = 0, \qquad \psi|_s = 0. \tag{4}$$

Here $\mathcal{L}$ and f do not depend on t.

Assume that the nonstationary problem (3), (4) is solved in a space of functions $\{\psi\}$, and that all functions ψ from this space satisfy the condition

$$|\psi| \leq M < \infty, \tag{5}$$

where M is a constant independent of t. In this case the solution of problem (1), (2) can be obtained with the aid of the operator S^T

$$\varphi^T = S^T \psi = \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} \psi \, dt, \tag{6}$$

where T is a parameter associated with the interval $0 \leq t \leq T$. Let us apply this operator to equation (3) on the interval $(0, T)$. Then we obtain

$$S^T \partial^2 \psi / \partial t^2 + \mathcal{L} \varphi^T = f. \tag{7}$$

Hence, using (4), we have

$$\mathcal{L} \varphi^T = f - \frac{2}{T^2} \psi(T). \tag{8}$$

Theorem. *If the linear nonsingular operator $\mathcal{L}$ is such that the solution of (3), (4) exists and satisfies condition (5), then the solution φ^T of problem (7), (4) converges in the norm of the space $\mathcal{L}_2$ to the solution of the stationary problem (1), (2) with rate $O(1/T^2\delta)$, where $\delta^2 > 0$ is the minimal eigenvalue of the operator $\mathcal{L}^* \mathcal{L}$.*

2°. Let us consider a difference analogue of this method. Suppose we have a discret-

boundary-value problem corresponding to problem (1), (2), i.e.

$$\Lambda \varphi_h = f_h, \tag{9}$$

$$\varphi_h \big|_{s_h} = 0, \tag{10}$$

where Λ is a difference operator approximating $\mathcal{L}$, and φ_h, f_h are grid functions. Instead of this difference problem, consider the following problem:

$$(\psi_h^{j+1} - 2\psi_h^j + \psi_h^{j-1})/\tau^2 + \Lambda \psi_h^j = f_h, \tag{11}$$

$$\psi_h^0 = \psi_h^1 = 0. \tag{12}$$

In what follows, for simplicity of notation we shall omit the index at ψ. It is assumed that, by the choice of the parameter τ, the existence of a bounded solution of the difference problem (11), (12) is ensured. We write the difference analogue M^T of the operator S^T in the form

$$\varphi_h^T = M^T\psi^j = \frac{2\tau^2}{T^2}\sum_{i=1}^{n}\sum_{j=1}^{i}\psi^j, \tag{13}$$

where $T = n\tau$.

Acting with the operator M^T on equation (11), we obtain

$$\Lambda\varphi_h^T = f_h - \frac{2}{T^2}\psi^{n+1}. \tag{14}$$

Comparing (14) with expression (9), by virtue of the linearity of the operator Λ, we have

$$\Lambda(\varphi_h - \varphi_h^T) = \Lambda e^T = \frac{2}{T^2}\psi^{n+1}.$$

Hence it is easy to obtain

$$\|e^T\| \leq M/T^2\delta,$$

where $\|\cdot\|$ is the Hilbert norm, $M = \text{const} < \infty$, and $\delta^2 > 0$ is the minimal eigenvalue of the difference operator $\Lambda^*\Lambda$.

3°. Consider the two simplest schemes for solving the equation, where Λ is a positive definite real operator.

Scheme I.

$$(\varphi^{j+1} - \varphi^j)/\tau + \Lambda\varphi^j = f, \qquad \varphi^0 = 0. \tag{15}$$

Scheme II.

$$(\varphi^{j+1} - 2\varphi^j + \varphi^{j-1})/\xi^2 + \Lambda\varphi^j = f, \qquad \varphi^0 = \varphi^1 = 0. \tag{16}$$

Here τ and ξ are, for the time being, arbitrary parameters, which will subsequently be chosen respectively from the condition of fastest convergence for Scheme I and stability for Scheme II.

If we sum (16) twice and multiply by some constant α_n, we obtain

$$\alpha_n\sum_{k=1}^{n}\frac{\varphi^{k+1} - \varphi^k}{\xi^2} + \Lambda\left[\alpha_n\sum_{k=1}^{n}\sum_{j=1}^{k}\varphi^j\right] = \alpha_n\sum_{k=1}^{n}kf. \tag{17}$$

Putting

$$\alpha_n = 1\Big/\sum_{k=1}^{n} k, \qquad \Phi_n = \alpha_n \sum_{k=1}^{n}\sum_{j=1}^{k} \varphi^j,$$

from (17) we have

$$\Lambda\Phi_n = f - \frac{\alpha_n}{\xi^2}\varphi^{n+1}. \tag{18}$$

The completeness of the system of eigenfunctions of the operator Λ is assumed. Next we carry out the Fourier analysis of problem (15). Then we obtain the relation–

for the expansion coefficients

$$\varphi_s^{j+1} = (1-\tau\lambda_s)\varphi_s^j + \tau f_s, \quad s = 1, 2, \ldots, s,$$

where λ_s are the eigenvalues of the operator Λ (assumed to be real), and the Fourier coefficients are determined with the involvement of the eigenfunctions of the adjoint equation. It is known that the optimal parameter τ, independent of the step, has the form

$$\tau = 2/(\Delta + \delta),$$

where Δ and δ are respectively the maximal and minimal eigenvalues of the operator Λ. Analogously, for the stability of scheme (16) we have

$$\xi^2\Delta < 2. \tag{19}$$

Consequently, one may set

$$\xi^2 = 2/(\Delta + \delta) < 2/\Delta.$$

From (18) we also obtain

$$\lambda_s\Phi_{ns} = f_s - \frac{\alpha_n}{\xi^2}\varphi_s^{n+1}. \tag{20}$$

If condition (19) is satisfied, then by methods of a priori estimates one can show that

$$|\varphi_s^{n+1}| \leqslant 2|f_s|/\lambda_s. \tag{21}$$

Indeed, for the Fourier coefficients of problem (16) we shall have

$$\varphi_s^{j+1} + \varphi_s^{j-1} - (2 - \xi^2\lambda_s)\varphi_s^j = \xi^2 f_s.$$

Multiply both sides of this expression by $\varphi_s^{j+1} - \varphi_s^{j-1}$, sum over j from 1 to n, and then use the inequality

$$\pm ab \leqslant \frac{1}{2\eta}a^2 + \frac{\eta}{2}b^2,$$

where $\eta > 0$ is a number. Then, when (19) is satisfied, we obtain relation (21). Further, under condition (21), from (20) we obtain

$$\Phi_{ns} = \frac{f_s}{\lambda_s}\left(1 - \frac{\alpha_n}{\xi^2}q\lambda_s^{-1}\right), \quad |q| \leqslant 2.$$

In the case of scheme I we have

$$\varphi_s^n = \frac{f_s}{\lambda_s}[1 - (1 - \tau\lambda_s)^n].$$

Let us write the expressions for the errors of the schemes:

$$R_I = \max_s (1 - \lambda_s\tau)^n, \quad R_{\text{II}} = \max_s \frac{2\alpha_n}{\xi^2\lambda_s}.$$

The greatest deviation occurs for $\lambda_s = \delta$; taking into account the values obtained earlier for τ and ξ^2, we get:

$$R_I = \left(\frac{\Delta - \delta}{\Delta + \delta}\right)^n, \quad R_{\text{II}} = \alpha_n\frac{\Delta + \delta}{\delta}.$$

Since $\alpha_n < 2/n^2$, with good accuracy

$$R_{\text{II}} = \frac{2}{n^2}\frac{\Delta + \delta}{\delta}.$$

Let $R_I = R_{\text{II}} = \varepsilon$ be the error. We find the necessary number of iterations n for each of the schemes:

$$n_I = \frac{1}{2}(p+1)|\ln\varepsilon|, \quad n_{\text{II}} = \sqrt{2(p+1)/\varepsilon},$$

where $p = \Delta/\delta$ is the Todd number. Hence it is clear that for large Todd numbers the two-step iteration method with averaging becomes more effective.

In a manner analogous to that described above, one can introduce for consideration a multistep iteration process with averaging, which is a difference analogue of a problem of the form

$$\partial^m \psi / \partial t^m + \mathcal{L}\psi = f$$

under the condition

$$\psi_t^{(k)}\big|_{t=0} = 0 \quad (k = 0, 1, 2, \dots, m-1), \qquad \varphi\big|_{s=0} = 0.$$

In this case the averaged solution of the problem

$$\varphi^T = S^T \psi = \frac{m!}{T^m} \int_0^T dt_1 \int_0^{t_1} \cdots \int_0^{t_{m-1}} \psi \, dt$$

in the class of bounded solutions ψ tends to the solution of problem (1), (2) with speed $O(1/T^m \delta)$.

Of interest is the problem of using an equation (and the corresponding difference analogues) of the form

$$\sum_{k=1}^{m} a_k \frac{\partial^k \psi}{\partial t^k} + \mathcal{L}\psi = f$$

with the limiting conditions indicated above. Here the a_k are as yet arbitrary coefficients, whose choice must be made in such a way that the solution ψ of the problem belongs to the class of bounded functions.

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Note: Figure translations are in progress. See original paper for figures.