ON TESTS OF A GENERAL LINEAR HYPOTHESIS WITH UNKNOWN WEIGHTS OF OBSERVATIONS

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Abstract

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ON TESTS OF A GENERAL LINEAR HYPOTHESIS WITH UNKNOWN WEIGHTS OF OBSERVATIONS

(Presented by Academician Yu. V. Linnik on 30 III 1967)

In the book of Yu. V. Linnik ([1]), properties are indicated that characterize similar Scheffé-type tests as Neyman structures. Such tests have hitherto been known only in the Behrens–Fisher problem. The conditions found are satisfied, for example, by all tests based on the t-ratio, in particular by the well-known test of E. Barankin ([3]) for one problem on linear regression. In ([1]) the question is posed of extending the results obtained to the case of a general linear hypothesis with unknown weights of observations. The present note serves this purpose.

Let $X_1, \dots, X_N$, where $X_i \in N(a_i, \sigma_i^2)$, be a repeated sample, and suppose that

$$a_i = \sum_1^s \alpha_{ij}\beta_j$$

(β_j are new parameters, and the matrix $\alpha = \|\alpha_{ij}\|$ is assumed given). Suppose that the σ_i^2, generally speaking, are different. The general linear hypothesis H_0 is the assertion

$$\beta_1 = B_1, \dots, \beta_r = B_r,$$

where B_i $(i = 1, \dots, r)$ are given constants.

Denote by $\mathfrak{X}$ the space generated by stochastically independent linear forms $l_1, \dots, l_\mu$ and $L_1, \dots, L_\nu$ such that

$$E\{l_i \mid H_0\} = E\{L_j\} = 0^* \quad (i = 1, \dots, \mu;\ j = 1, \dots, \nu),$$
$$D\{l_i\}/D\{l_1\} = C_i \quad (i = 1, \dots, \mu),$$

and $C_i > 0$ does not depend on $\sigma = (\sigma_1, \dots, \sigma_N)$, whereas the forms L_j may have different variances.

Consider two continuous homogeneous functions T_1 and T_2 of degree $n > 0$ with the following properties:

$$T_1(l_1, \dots, l_\mu) > 0, \quad \text{if } (l_1, \dots, l_\mu) \neq (0, \dots, 0),$$
$$T_2(L_1, \dots, L_\nu) > 0, \quad \text{if } (L_1, \dots, L_\nu) = (0, \dots, 0),$$

and

$$T_2 > C > 0, \tag{1}$$

if at least one of its arguments is equal to one. Then the following is valid.

Theorem. *If a similar test φ (generally speaking, randomized) is defined on the space $\mathfrak{X}$ and accepts the null hypothesis at least under the conditions*

$$T_1/T_2 \leqslant \varepsilon \tag{2}$$

($\varepsilon > 0$ sufficiently small) and

$$\sqrt{Q_1} = \left[\sum_1^\mu \frac{l_i^2}{c_i}\right]^{1/2} < \delta \tag{3}$$

* The mathematical expectations of the forms L_j are equal to zero for any values $\beta = (\beta_1, \dots, \beta_s)$.

($\delta > 0$ is any fixed number), then the variances of the linear forms $L_1, \dots, L_\nu$ satisfy the relations

$$D\{L_j\}/D\{l_1\} = d_j \qquad (j = 1, \dots, \nu), \tag{4}$$

and $d_j > 0$ do not depend on σ.

It follows immediately from this that φ is a Neyman structure for the exponential family generated by the forms $l_1, \dots, l_\mu, L_1, \dots, L_\nu$. For $\mu = 1$ and in the case of the Behrens–Fisher problem, we again obtain the result of Yu. V. Linnik ([1]).

As in ([1]), the proof is by contradiction and is based on consideration of the test-similarity condition

$$E_\sigma\{\varphi \mid H_0\} = \alpha \quad \text{uniformly in } \sigma. \tag{5}$$

Let us briefly explain the proof. Suppose that (4) holds for $j = 1, \dots, p < \nu$, and transform (5) as described in ([1]). As a result, σ are replaced by new parameters

θ, which we regard as complex. Then we consider a parametric point that is singular for both sides of the transformed relation (5). In a neighborhood of this point the stated relation has the form

$$\int_{\mathfrak{x}}\cdots\int \varphi\,\frac{dl_1\cdots dl_\mu\,dL_1\cdots dL_\nu}{Q^{\tau+(\mu+\nu)/2}}$$

$$= A_1 G_1(\tau)\xi^{-\tau}(i\xi)^{-\tau-(\nu-p)/2}\prod_{p+1}^{\nu}\left[d_{jN-1}i\xi(d_{jN}-d_{jN-1})\right]^{1/2}, \qquad (6)$$

where

$$Q = Q_1 + \sum_1^p \frac{L_j^2}{d_j} + \sum_{p+1}^{\nu} L_j^2 \frac{i\xi}{d_{jN-1}i\xi + (d_{jN}-d_{jN-1})} + i\xi^2;$$

A_1 is a positive constant; $G_1(\tau)$ is a function regular in $\mathrm{Re}(\tau) > 0$, and $\xi > 0$ is a number which we henceforth make arbitrarily small. We are now interested in the behavior of the moduli of both sides of (6) as $\xi \downarrow 0$. A lower estimate for the modulus of the right-hand side has the form

$$B\xi^{-2\tau-(\nu-p)/2},$$

where the symbol B denotes a bounded quantity. To obtain an upper estimate for the modulus of the left-hand side of (6), we divide the space $\mathfrak{x}$ into "layers" :

I. $2^{m-1}\xi \le \sqrt{Q_1} < 2^m\xi$,

II. $\frac{1}{2^m}\xi \le \sqrt{Q_1} < \frac{1}{2^{m-1}}\xi$ inside (3),

III. $2^{m-1}\delta \le \sqrt{Q_1} < 2^m\delta$ in the remaining part of the space $\mathfrak{x}$.

The estimate is carried out separately in each "layer." In doing so, the properties of the functions T_1 and T_2 and the inequalities

$$|Q| \ge |\,\mathrm{Re}\,Q| \ge Q_1, \qquad |Q| \ge |\,\mathrm{Im}\,Q| \ge B\xi^2$$

are used for $|L_j| < \xi$. In the end we obtain the desired estimate

$$B\xi^{-2\tau}.$$

To avoid a contradiction in the behavior of both sides of (6), we must have $\nu - p = 0$.

Since the exponential family generated by the forms $l_1, \ldots, l_\mu, L_1, \ldots, L_\nu$ is one-parametric under H_0, it follows from the Lehmann–Scheffé theory that φ is a Neyman structure with respect to the statistic

$$\sum_1^\mu \frac{l_i^2}{\sigma_i} + \sum_1^\nu \frac{L_j^2}{d_j}.$$

The forms l_i are naturally to be chosen in such a way that the indicated statistic is not sufficient under the specified alternative H_1; then the test φ is nontrivial.

In many practical cases the sample consists of m independent subsamples of sizes n_k $(k = 1, \ldots, m)$, drawn respectively from

$$N(a_k, \sigma_k^2).$$

Then $N = \sum_1^m n_k$, and μ, ν must satisfy the condition

$$\mu + \nu \leq \min n_k.$$

The optimal choice of the forms $l_1, \ldots, l_\mu, L_1, \ldots, L_\nu$ depends on the particular form of the functions T_1 and T_2 and can be carried out, for example, on the basis of the same considerations as in (2) or (3), if tests based on the F-distribution are constructed.

The result obtained admits a multivariate generalization. In this case T_1 and T_2 are replaced respectively by the matrices Q_1 and Q_2. Instead of inequality (2), acceptance of the null hypothesis is postulated under the condition

$$\mathrm{sp}\, Q_1 Q_2^{-1} \leq \varepsilon.$$

Condition (1) takes the form

$$|Q_2| > c > 0,$$

if all entries of the matrix Q_2 are bounded.

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