ON TOPOLOGICAL EQUIVALENTS OF THE BRANCHING HYPOTHESIS\

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Abstract

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ON TOPOLOGICAL EQUIVALENTS OF THE BRANCHING HYPOTHESIS*

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1. The aim of the present paper is to strengthen the results of our preceding paper ([1]), consisting in the finding of new topological equivalents of the set-theoretic condition (α) for a strongly inaccessible cardinal number, which we call the branching hypothesis for this number.

Below, as in ([1]), $\aleph_\sigma$ is strongly inaccessible and $\mathfrak{m}$ is an infinite cardinal number. We take the remaining terms and notation from ([1]) without qualification, and add the following new ones. A system of sets Σ is called an **open $\mathfrak{m}$-subbase** of a topological space if the collection of intersections of all subsystems in Σ of cardinality $< \mathfrak{m}$ forms an open base of this space. A partially ordered set M is called **$\mathfrak{m}$-directed** if, for each of its subsets of cardinality $< \mathfrak{m}$, there exists an upper bound in M. In particular, $\aleph_0$-directed sets are directed sets in the usual sense of the word. If X is a topological space, then $\exp(\mathfrak{m}, X)$ denotes the space on the family of all nonempty closed subsets of X, whose topology is given by the open base consisting of sets of the form $B(G_0, \mathfrak{G})$, where, for an open G_0 and a family $\mathfrak{G}$ of cardinality $< \mathfrak{m}$ of open sets in X,

$$B(G_0, \mathfrak{G}) = \{F \mid F \subseteq G_0,\ \forall G \in \mathfrak{G},\ F \cap G \supset \Lambda\}.$$

In particular,

$$B(G_0, \{G_1, \dots, G_n\}) = B(G_0, G_1, \dots, G_n)$$

from ([2]), p. 168, and for $\mathfrak{m} = \aleph_0$ we have the usual Vietoris exponential space (see, for example, ([2]) or ([3])).

We formulate the following properties of the number $\aleph_\sigma$, supplementing the list from ([1]):

$\gamma_T(\gamma^+)$. The Tikhonov ($\mathcal{T}_\sigma-$) product of $\aleph_\sigma$-bicompact spaces of weight $\leq \aleph_\sigma$, taken in number $\leq \aleph_\sigma$, is $\aleph_\sigma$-bicompact (analogues of Tikhonov' s theorem).

δ. The assertion of Theorem 1 from ([1]).

δ^+. If from every cover of a space X by sets of some open $\aleph_\sigma$-subbase of it of cardinality $\aleph_\sigma$ one can extract a subcover of cardinality $< \aleph_\sigma$, then X is $\aleph_\sigma$-bicompact (an analogue of Aleksandrov' s lemma).

ε. If X is an $\aleph_\sigma$-bicompact space of weight $\aleph_\sigma$, then $\exp(\aleph_\sigma, \mathcal{T}_\sigma X)$ is also an $\aleph_\sigma$-bicompact space of weight $\aleph_\sigma$ (an analogue of Vietoris's theorem, ([3]), p. 161).

ζ. If $\{X_\lambda, \pi_\mu^\lambda \mid \mathrm{card}\{\lambda\} = \aleph_\sigma\}$ is an inverse $\aleph_\sigma$-directed spectrum of nonempty $\aleph_\sigma$-bicompact $\mathcal{T}_2$-spaces of weight $\leq \aleph_\sigma$, then its limit is nonempty (an analogue of Steenrod' s theorem ([4]), p. 51).

Theorem. *For strongly inaccessible $\aleph_\sigma$, the conditions β, β^+, γ, γ_T, γ^+, δ, δ^+, ε, and ζ are equivalent to the branching hypothesis α.*

Since the equivalence of α, β, β^+, γ, and δ was proved in ([1]), it is necessary to prove only the remaining implications. Obviously, $\delta^+ \Rightarrow \delta$, and the implication $\alpha \Rightarrow \delta^+$ can be proved without changing anything in the method of proof of $\alpha \Rightarrow \delta$

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(theorem 1 of ([1])). We shall derive the rest according to the scheme: $\delta^+ \Rightarrow \varepsilon \Rightarrow \gamma \Rightarrow \alpha \Rightarrow \delta^+ \Rightarrow \nu^\tau \Rightarrow \nu_\tau \Rightarrow \xi \Rightarrow \alpha$. In the proof of $\delta^+ \Rightarrow \varepsilon$ we used Frink' s idea (see, for example, ([3]), p. 161), and in the proof of $\varepsilon \Rightarrow \gamma$ the idea of Maryanovich ([5]) in similar situations.

2. Let X be a topological space; F_0, F_1 closed in X; put

$$\Phi(F_0, F_1) = \{F \mid F \cap F_0 = \Lambda \text{ or } \Lambda \subset F \subset F_1\},$$

where F is closed in X. A system of sets shall be called $\mathfrak{m}$-additive ($\mathfrak{m}$-multiplicative) if it contains all unions (intersections) of its subsystems of cardinality less than $\mathfrak{m}$.

(a) If $F_0' \cap F_1'' = \Lambda$, then

$$\Phi(F_0', F_1') \cap \Phi(\Lambda, F_1'') = \Phi(\Lambda, F_1' \cap F_1'').$$

Let F belong to the left-hand side; then $F \in \Phi(\Lambda, F_1'')$, whence $F \subset F_1''$; since $F_0' \cap F_1'' = \Lambda$, $F \cap F_0' = \Lambda$. But $F \in \Phi(F_0', F_1')$, whence $F \subset F_1'$. Thus $F \subset F_1' \cap F_1''$, and F belongs to the right-hand side. The reverse inclusion is obvious. It is also obvious that

(b)

$$\bigcap_\lambda \Phi(\Lambda, F_1^\lambda) = \Phi\left(\Lambda, \bigcap_\lambda F_1^\lambda\right).$$

(c) Let $\mathfrak{m}$ be regular and X an arbitrary $\mathfrak{m}$-bicompact space. Then any $\mathfrak{m}$-centered system of families of closed subsets of X of the form $\Phi(F_0, F_1)$ has nonempty intersection.

Let $\widetilde{\Phi}$ be our system. Extend $\widetilde{\Phi}$ to $\widetilde{\Phi}^{+}$ by adding all sets of the form $\Phi(F_0, F_1)$ which are intersections of subsystems in $\widetilde{\Phi}$ of cardinality $< \mathfrak{m}$. Then, by regularity of $\mathfrak{m}$, $\widetilde{\Phi}^{+}$ contains all $\Phi(F_0, F_1)$ which are intersections of subsystems in $\widetilde{\Phi}^{+}$ of cardinality $< \mathfrak{m}$ (in particular, $\Phi(\Lambda, X)$, the intersection of the empty subsystem) and does not contain the empty family $\Lambda = \Phi(\Lambda, \Lambda)$. We shall prove that even $\bigcap \widetilde{\Phi}^{+} \supset \Lambda$. Let

$$\widetilde{\Phi}^{+} = \{\Phi(F_0^{\alpha}, F_1^{\alpha})\}.$$

Let Ψ be the family of those F_1^{α} for which $F_0^{\alpha} = \Lambda$. Then, by (b), Ψ is $\mathfrak{m}$-multiplicative. Put $F = \bigcap \Psi$ and prove that $F \in \bigcap \widetilde{\Phi}^{+}$, i.e. $\forall \alpha\ F \in \Phi(F_0^{\alpha}, F_1^{\alpha})$. Fix α; if $F_0^{\alpha} \cap F \supset \Lambda$, then $F \subset \Phi(F_0^{\alpha}, F_1^{\alpha})$, so let further $F_0^{\alpha} \cap F = \Lambda$. We first prove that there exists $F_1^{\alpha'} \in \Psi$ such that $F_0^{\alpha} \cap F_1^{\alpha'} = \Lambda$; for this suppose the contrary: for all $F_1^{\alpha'} \in \Psi$, $F_0^{\alpha} \cap F_1^{\alpha'} \supset \Lambda$. Then the system

$$\{F_0^{\alpha} \cap F_1^{\alpha'} \mid F_1^{\alpha'} \in \Psi\}$$

is $\mathfrak{m}$-centered, since Ψ is $\mathfrak{m}$-multiplicative. But then

$$\bigcap \{F_0^{\alpha} \cap F_1^{\alpha'} \mid F_1^{\alpha'} \in \Psi\} = F_0^{\alpha} \cap F \subset \Lambda$$

by $\mathfrak{m}$-bicompactness of X, which contradicts the initial condition. Thus there exists $F_1^{\alpha'}$ in Ψ such that

$$F_0^{\alpha} \cap F_1^{\alpha'} = \Lambda.$$

Then by (a)

$$\Phi(\Lambda, F_1^{\alpha} \cap F_1^{\alpha'}) = \Phi(\Lambda, F_1^{\alpha'}) \cap \Phi(F_0^{\alpha}, F_1^{\alpha}),$$

and the sets on the right belong to $\widetilde{\Phi}^{+}$, whence $\Phi(\Lambda, F_1^{\alpha} \cap F_1^{\alpha'}) \in \widetilde{\Phi}^{+}$, $F_1^{\alpha} \cap F_1^{\alpha'} \in \Psi$, $F = \bigcap \Psi \subset F_1^{\alpha} \cap F_1^{\alpha'} \subset F_1^{\alpha}$. Since Ψ is $\mathfrak{m}$-multiplicative and does not contain the empty set, Ψ is $\mathfrak{m}$-centered in the $\mathfrak{m}$-bicompact space X, so that $F_1^{\alpha} \supset F = \bigcap \Psi \supset \Lambda$ and $F \in \Phi(F_0^{\alpha}, F_1^{\alpha})$.

(d) If $\mathfrak{B}$ is an $\mathfrak{m}$-additive open base of the $\mathfrak{m}$-bicompact space X, then

$$\{B(H_0, \mathfrak{H}) \mid H_0 \in \mathfrak{B},\ \mathfrak{H} \subseteq \mathfrak{B},\ \operatorname{card} \mathfrak{H} < \mathfrak{m}\}$$

is an open base in $\exp(\mathfrak{m}, X)$.

Let $B(G_0, \mathfrak{G})$ be a neighborhood of F in $\exp(\mathfrak{m}, X)$. Take a covering of F by sets from $\mathfrak{B}$ lying in G_0, and select from it a subcover of cardinality $< \mathfrak{m}$; the union of the latter, by $\mathfrak{m}$-additivity of $\mathfrak{B}$, is some $H_0 \in \mathfrak{B}$, and $F \subseteq H_0 \subseteq G_0$. Further, for each $G \in \mathfrak{G}$ take $x(G) \in G \cap F$ and in $\mathfrak{B}$ choose $H(G)$, $x(G) \in H(G) \subseteq G$. Denote the collection of all such $H(G)$ by $\mathfrak{H}$. Then

$$F \in B(H_0, \mathfrak{H}) \subseteq B(G_0, \mathfrak{G}),$$

as was required here.

(e) If X has weight $\aleph_\sigma$, then $\exp(\aleph_\sigma, X)$ has weight $\geq \aleph_\sigma$.

Since X has weight $\aleph_\sigma$, there are at least $\aleph_\sigma$ closed sets in X. But from $F_1 \setminus F_2 \supset \Lambda$ it follows that

$$F_1 \in B(X, CF_2) \setminus B(X, CF_1),$$

so that there are at least $\aleph_\sigma$ open sets in $\exp(\aleph_\sigma, X)$. And a space of weight $\aleph_\nu < \aleph_\sigma$ has $\leq 2^{\aleph_\nu} < \aleph_\sigma$ open sets.

From (d) and (e) there follows, independently of the branching hypothesis, the assertion part of (ε):

(f) If X is an $\aleph_\sigma$-bicompact space of weight $\aleph_\sigma$, then $\exp(\aleph_\sigma, X)$ has weight $\aleph_\sigma$.

Since $\sum_{\alpha<\sigma} \aleph_\sigma^{\aleph_\alpha} = \aleph_\sigma$, for the proof it is enough to enlarge an open base of X of cardinality $\aleph_\sigma$ to an equipotent $\aleph_\sigma$-additive one by adding the unions of all its subsystems of cardinality $< \aleph_\sigma$.

(g) $\bigcap\limits_\lambda B(G_0^\lambda, \mathfrak{G}^\lambda) = B\left(\bigcap\limits_\lambda G_0^\lambda, \bigcup\limits_\lambda \mathfrak{G}^\lambda\right)$; in particular, $\bigcap\limits_\lambda B(G_0, G_1^\lambda) = B(G_0, \{G_1^\lambda\})$.

3. $(\delta^+) \Rightarrow (\varepsilon)$. Let X be an $\aleph_\sigma$-bicompact space of weight $\aleph_\sigma$; by (δ), $\mathcal{T}^\sigma X$ is also an $\aleph_\sigma$-bicompact space of weight $\aleph_\sigma$, and let $\mathfrak{B}$ be an $\aleph_\sigma$-additive open base of $\mathcal{T}^\sigma X$ of cardinality $\aleph_\sigma$ (cf. the proof of (f)). Then, by (d), $\{B(H_0, \mathfrak{H}) \mid H_0 \in \mathfrak{H},\ \mathfrak{H} \subseteq \mathfrak{B},\ \operatorname{card} \mathfrak{H} < \aleph_\sigma\}$ is an open base of $\exp(\aleph_\sigma, \mathcal{T}^\sigma X)$ of cardinality $\aleph_\sigma$. Since $\mathcal{T}^\sigma X$ is a $\mathcal{T}^\sigma$-space, by (g) the family $\{B(H_0, H_1) \mid H_0, H_1 \in \mathfrak{B}\}$ forms an open $\aleph_\sigma$-subbase of $\exp(\aleph_\sigma, \mathcal{T}^\sigma X)$ of cardinality $\aleph_\sigma$, and by (δ^+), passing to complements $CB(H_0, H_1) = \Phi(CH_0, CH_1)$, and using (c), we obtain the required assertion.

$(\varepsilon) \Rightarrow (\gamma)$. Replacing, for the disjoint sum $\cup$, the sign Π, introduce the topological sum $\Pi_\lambda D_\lambda = S$, where the D_λ are two-point discrete spaces, $\operatorname{card}\{\lambda\} = \aleph_\sigma$, and consider its one-point extension $S^+ = S\Pi\infty$, where neighborhoods of ∞ are taken to be $O(\infty) = \infty\Pi M$, $M \subseteq S$, $\operatorname{card} CM < \aleph_\sigma$. It is obvious that a set is closed in S^+ if and only if it either contains ∞, or has cardinality $< \aleph_\sigma$. It is also easy to see that S^+ is an $\aleph_\sigma$-bicompact $\mathcal{T}^\sigma$-space of weight $\aleph_\sigma$, and, by (ε), $\exp(\aleph_\sigma, S^+)$ has the same properties. Consider in $\exp(\aleph_\sigma, S^+)$ points of the form $\Pi_\lambda x_\lambda \Pi\infty$, where $x_\lambda = x(\lambda)$ is a choice function for the whole family $\{D_\lambda\}$; denote all such points by Ξ and prove that Ξ is closed in $\exp(\aleph_\sigma, S^+)$. Let $F \in C\Xi$; then for F there are two possibilities: (1) $\exists\lambda_0$, $F \cap D_{\lambda_0} = \Lambda$; then $B(CD_{\lambda_0}, CD_{\lambda_0})$ is a neighborhood of F not meeting Ξ; (2) $\exists\lambda_0$, $F \supset D_{\lambda_0}$, $D_{\lambda_0} = \{x_{\lambda_0}, y_{\lambda_0}\}$; then $B(S^+, \{x_{\lambda_0}\}, \{y_{\lambda_0}\})$ is again a neighborhood of F not meeting Ξ. Since S^+ is $\aleph_\sigma$-bicompact, Ξ is also $\aleph_\sigma$-bicompact. But the correspondence $\Pi_\lambda x_\lambda \Pi\infty \to \{x_\lambda\}$ is a natural homeomorphism of Ξ onto the $\mathcal{T}^\sigma$-product of the family $\{D_\lambda\}$, whence everything follows.

$(\delta^+) \Rightarrow (\gamma^+) \Rightarrow (\gamma_T)$. Let $P = \Pi^\sigma X_\lambda$ be our $\mathcal{T}^\sigma$-product; obviously, it is enough to prove the $\aleph_\sigma$-bicompactness of the product $P^\sigma = \Pi^\sigma \mathcal{T}^\sigma X_\lambda$ with the stronger topology. By (δ), the $\mathcal{T}^\sigma X_\lambda$ are also $\aleph_\sigma$-bicompact and have weight $\leq \aleph_\sigma$. Let π_λ be the projection of P^σ onto $\mathcal{T}^\sigma X_\lambda$, and let $\mathfrak{B}_\lambda$ be open bases in $\mathcal{T}^\sigma X_\lambda$ of cardinality $\leq \aleph_\sigma$. Since the $\mathcal{T}^\sigma X_\lambda$ are $\mathcal{T}^\sigma$-spaces, $\{\pi_\lambda^{-1}(B_\lambda) \mid B_\lambda \in \mathfrak{B}_\lambda,\ \lambda \in \{\lambda\}\}$ is an open $\aleph_\sigma$-subbase of P^σ (of cardinality $\leq \aleph_\sigma$). The rest of the argument is carried out analogously to ([6]), p. 143, with the obvious modification. Since the Tikhonov product is a compactification of the corresponding $\mathcal{T}^\sigma$-product, $(\gamma^+) \Rightarrow (\gamma_T)$.

$(\gamma_T) \Rightarrow (\zeta)$. Let $\{X_\lambda, \pi_\mu^\lambda\}$ be our spectrum. The $\aleph_\sigma$-centeredness of the system $\{S_\mu^\lambda \mid \lambda > \mu\}$, $S_\mu^\lambda = \{x \mid x \in \Pi X_\lambda,\ \pi_\mu^\lambda(x_\lambda) = x_\mu\}$, is proved analogously to the corresponding centeredness in ([4]), p. 51. Since ΠX_λ is $\aleph_\sigma$-bicompact, $\bigcap_{\lambda>\mu} S_\mu^\lambda \supset \Lambda$, as was required.

$(\zeta) \Rightarrow (\alpha)$. Let S be a branching system of order ω_σ. Let $x \in S$; denote by $p(x)$ the order of x in S and put

$$S_\lambda = \{x \mid x \in S,\ p(x) = \lambda\}, \qquad \tau(x) = \sup\{p(y) \mid y \geq x \text{ in } S\}.$$

It is easy to verify the following properties of $\tau(x)$: (1) if $y < x$ in S, then $\tau(y) \geq \tau(x)$, and (2) if $\tau(y) \geq \lambda$, $y \in S_\mu$, $\mu < \lambda$, then $\tau(y) = \sup\{\tau(v) \mid v \in S_\lambda,\ v \geq y \text{ in } S\}$. Put $S_\lambda^0 = \{x \mid x \in S_\lambda, \tau(x) = \omega_\sigma\}$ and prove that all these sets are nonempty. Indeed, otherwise there exists $\lambda_0 <$

$< \omega_\sigma$ such that for all $x \in S_{\lambda_0}$, $\tau(x) < \omega_\sigma$. Since $\operatorname{card} S_{\lambda_0} < \aleph_\sigma$, in view of the regularity of ω_σ we have $\sup\{\tau(x) \mid x \in S_{\lambda_0}\} < \omega_\sigma$, and the order of the entire system is $< \omega_\sigma$. We now define, for $\lambda > \mu$, $x \in S_\lambda^0$, $\pi_\mu^\lambda x = y$, where $y \in S_\mu$ and $y \leq x$ in S. Such a y exists and is unique by the definition of a branching system. By property (1), for $\tau(x)$ the mapping π_μ^λ maps S_λ^0 into S_μ^0, and we shall prove that this mapping is onto.

Suppose the contrary: there exists $z \in S_\mu^0$, $z \notin \pi_\mu^\lambda(S_\lambda^0)$. Since $\tau(z) = \omega_\sigma > \lambda$, by property (2)

$$\tau(z) = \sup\{\tau(v) \mid v \in S_\lambda \setminus S_\lambda^0,\ v \geq z \text{ in } S\}.$$

Since all $\tau(v) < \omega_\sigma$ and $\operatorname{card} S_\lambda < \omega_\sigma$, we have $\tau(z) < \omega_\sigma$, which is impossible. Considering now all S_λ^0 with the discrete topologies, we obtain an inverse spectrum $\{S_\lambda^0, \pi_\mu^\lambda\}$ satisfying condition (ζ). An element of its nonempty limit gives the sequence in S required by condition (α).

Remark. Since assertion (c) of item 2 is proved without using the axiom of choice, Vietoris' theorem on the bicompactness of the ordinary exponential space over a bicompact space is obtained by using only Alexander' s lemma. Since, by Marjanović' s method, the Tikhonov product $\prod_\lambda X_\lambda$ of bicompact $\mathcal{T}_2$-spaces is embedded as a closed set in the exponential space over the one-point bicompactification of P. S. Aleksandrov of the topological sum $\coprod_\lambda X_\lambda$, it follows from Alexander' s lemma that Tikhonov' s theorem already holds for $\mathcal{T}_2$-multipliers. This method does not work in the case of $\mathcal{T}_2$-multipliers, since then the corresponding set in the exponential space ceases to be closed, which was to be expected, since according to Kelley' s theorem (7) the case of $\mathcal{T}_1$-multipliers is already exactly equivalent to the axiom of choice (cf. the comments in this connection in (8), p. 274).

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Note: Figure translations are in progress. See original paper for figures.