AN ANALOG OF THE TOPOLOGY OF A SPACE OF MEASURABLE FUNCTIONS IN A K-LINEAL WITH UNIT

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Abstract

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AN ANALOG OF THE TOPOLOGY OF A SPACE OF MEASURABLE FUNCTIONS IN A K-LINEAL WITH UNIT

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1. A fundamental system of neighborhoods of zero of a linear topological space X is called a basis. If the space X is separated, is a K-lineal, and has a basis consisting of normal ([1]) neighborhoods, then we shall call the basis normal, and X a KT-lineal. By convergence of a net $x_\alpha \to x$ is meant topological convergence. A KT-lineal has the basic properties of a KN-lineal ([1]). A topologically countably complete KT-lineal with unit (b) is complete in the sense of ([2–4]) (with respect to the norm of bounded elements).

2. Here an analog of the topology of a space of measurable functions is constructed in a certain foundation ([1]) Y of an Archimedean K-lineal Z with unit that is complete in the sense of ([2–4]) (an a.p.e. K-lineal). Conditions are found under which $Y = Z$. The basis of the KT-lineal is constructed essentially in the same way as was done in ([5]) for the universal semiring, i.e. an extended K-space, but under weaker restrictions on the K-lineal Z and on the topology of its basis I. The topological continuity of the mapping of a KT-lineal X, structurally embedded in Z, is established when the topologies coincide in $X \cap I$.

3. Suppose that in some K-lineal Z there is distinguished a certain system of normal sets W, satisfying the first axiom of separatedness at zero: $\cap W = \{0\}$, and the requirements imposed on a basis of a linear topological space (see, for example, Theorem 1 (1.XI) in ([6])), except for one: some of the sets W may not absorb any elements in Z. Then Z shall be called a generalized KT-lineal with basis $\{W\}$.

If some normal sublineal Y of the generalized KT-lineal Z with unit and basis $\{W\}$ contains its unit and is a KT-lineal with basis $\{W \cap Y\}$, then Y shall be called a KT-foundation in Z.

If each set W absorbs the unit of the generalized KT-lineal Z, then the sublineal

Z_0 of bounded elements is a KT-foundation in Z. It is clear that any KT-foundation $Y \supset Z_0$.

If each set W, moreover, absorbs any element of a certain linear substructure X of the K-lineal Z, then there exists a KT-foundation $Y \supset X + Z_0$.

4. An a.p.e. K-lineal (see above, item 2) Z is realized ([3,4]) as the K-lineal $Z(Q)$ of certain continuous functions, allowed to take infinite values, on a certain bicompact ([1]) Q. Within Z there is the K-lineal $Z_0 = C(Q)$ of all continuous bounded functions ([2]). The base I of unit elements e is structurally isomorphic to the system of open-closed sets $E_e \subset Q$.

Suppose there is a system of subsets $\Gamma \subset I$ satisfying the conditions:

$$(\Gamma 1) \quad \text{For any } \Gamma_i \text{ there exists } \Gamma \subset \Gamma_1 \cap \Gamma_2.$$

$$(\Gamma 2) \quad \text{For each } \Gamma_0 \text{ there is } \Gamma \text{ such that if } e_i \in \Gamma, \text{ then } e_1 \vee e_2 \in \Gamma_0.$$

$$(\Gamma 3) \quad \text{All } \Gamma \text{ are normally contained in } I.$$

$$(\Gamma 4) \quad \cap \Gamma = \{0\}.$$

Define in Z the sets $W_{\Gamma\varepsilon}$ ([5]): $z \in W_{\Gamma\varepsilon}$ if there exists an element $e \in \Gamma$ such that $\{t;\ |z(t)| \geq \varepsilon\} \subset E_e$ (here $0 < \varepsilon \leq 1$).

The following assertions hold.

a) In Z there is a KT-fundament Y with basis $\{Y \cap W_{\Gamma\varepsilon}\}$, for example $Y = Z_0$.

b) The system $\{\Gamma\}$ determines a topology of the basis I,

$$\Gamma = I \cap W_{\Gamma\varepsilon}.$$

The basis I is a (topological) subspace of the KT-lineal Y. Topological convergence of a direction $e_\alpha \to e$ in I means that the symmetric difference $e_\alpha Ce + eCe_\alpha \in \Gamma$ for $\alpha \geq \alpha_0$.

c) For any finite set of elements y_i of the KT-fundament Y and an arbitrary neighborhood Γ there exists an element $e \in \Gamma$ such that

$$\bigcup_i \{t;\ y_i(t) = \infty\} \subset E_e.$$

On the set $E_{Ce} = Q \setminus E_e$ the functions $y_i(t)$ are bounded.

d) In the KT-fundament Y with basis $\{Y \cap W_{\Gamma\varepsilon}\}$, an abstract function generated [4] in an a.p.e. K-lineal Z by a real continuous function of several variables is topologically continuous at all points in whose vicinity the abstract function has meaning in Y.

e) In order that a certain normal sublineal Y, containing the unit of the given K-lineal Z, be a KT-fundament with basis $\{Y \cap W_{\Gamma\varepsilon}\}$, it is necessary and sufficient that the following condition be satisfied:

e′) for every element $y \in Y$ and arbitrary neighborhood Γ there is an element $e \in \Gamma$ such that

$$\{t;\ y(t) = \infty\} \subset E_e.$$

5. Let now the bicompactum Q be completely disconnected [1]. A special case of an a.p.e. K-lineal $Z(Q)$ with a completely disconnected bicompactum Q is a K_σ-space [1] with a unit.

f) For every element $z \in Z$ and arbitrary quantities $\mu, \nu :\ \ \mu < \nu \leq +\infty$, there exists an element e such that

$$\{t;\ z(t) \geq \nu\} \subset E_e, \qquad z \geq \mu e.$$

g) Condition e′) is equivalent to the following relation between monotone (r)-convergence of a sequence of unit elements and topological convergence:

$$(\Gamma 5) \quad \text{If } e_n \geq e_{n+1} \xrightarrow{(r)} 0 \text{ in } Y, \text{ then } e_n \to 0 \text{ in } I.$$

If condition (Γ5) is satisfied for $Y = Z$, then Z is a KT-lineal with basis $\{W_{\Gamma\varepsilon}\}$.

h) In order that the whole K-lineal Z be a KT-lineal with basis $\{W_{\Gamma\varepsilon}\}$, it is sufficient that monotone (o)-convergence of a sequence in I imply topological convergence:

$$(A1) \quad \text{If } e_n \downarrow 0, \text{ then } e_n \to 0.$$

Condition (A1) is weaker than that indicated in axiom 6 on p. 52 in [5].

The topology defined by the basis $\{W_{\Gamma\varepsilon}\}$ or by its trace $\{Y \cap W_{\Gamma\varepsilon}\}$ is an analogue of the topology of the space of measurable functions; the sets E_e play the role of measurable sets in the realization of KB-spaces [7–10].

i) If from the convergence $e_\alpha \xrightarrow{(0)} 0$ $(\alpha \in A)$ it follows that $e_\alpha \to 0$, then from the convergence $y_\alpha \xrightarrow{(0)} y$ $(\alpha \in A)$ it follows that $y_\alpha \to y$.

6. Let a certain KT-lineal X with normal basis $\{V\}$ be a normal sublineal of some a.p.e. K-lineal Z. Put

$$\Gamma = V \cap I. \tag{1}$$

Then all conditions (Γ1—4) are fulfilled.

If the bicompactum Q is totally disconnected, then

$$\mu V \subset W_{\Gamma\varepsilon} \qquad (0 < \mu < \varepsilon);$$

in Z there is a KT-fundament $Y \supset X + Z_0$; the KT-lineal X with basis $\{V\}$ is topologically continuously embedded in the KT-lineal Y with basis $\{Y \cap W_{\Gamma\varepsilon}\}$.

7. Let some KT-lineal X with unit and normal basis $\{V\}$ be a normal sub-lineal of some **inner-normal** ([4]) a.p. K-lineal $Z(Q)$ with the same unit (a particular case of such a K-lineal Z is a K_σ-space with unit), and let the neighborhoods Γ be defined by equality (1). Then, in order that the mapping of the KT-lineal X with basis $\{X \cap W_{\Gamma\varepsilon}\}$ into the KT-lineal X with basis $\{V\}$ be topologically continuous, it is necessary and sufficient that the condition be fulfilled (see axiom 8 on p. 52 in ([5])):

(Γ6) For every neighborhood V_0 there exists a neighborhood Γ such that the algebraic product $\Gamma X \subset V_0$.

If, moreover, the bicompactum Q is totally disconnected, then the indicated mapping is topologically bicontinuous.

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Note: Figure translations are in progress. See original paper for figures.