A TRACE FORMULA IN THE THEORY OF GEODESICS

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Abstract

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A TRACE FORMULA IN THE THEORY OF GEODESICS

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With the system of Jacobi equations for geodesics one can naturally associate a self-adjoint differential operator. In the present note a formula, called the trace formula, is proved; it expresses the perturbation determinant of this operator through the determinant of a finite-dimensional operator—the differential of the exponential mapping.

Trace formula. On a complete Riemannian manifold M, consider a fixed geodesic $\Gamma : [0, 1] \to M$, parametrized proportionally to length. Denote the parameter by t. The Jacobi equation for a vector field $W = W(t)$ on Γ can be represented in the form

$$\mathcal{L}W \equiv -\frac{D^2}{dt^2}W + Q_{VW} = 0.$$

In the definition of the differential expression $\mathcal{L}W$, D/dt is the covariant derivative along Γ; $V = d\Gamma/dt$ is the velocity vector field; $Q_V = R(V, \cdot)V : TM_{\Gamma(t)} \to TM_{\Gamma(t)}$, and $R(V, W)$ is the curvature transformation (^{1}). On the set of complex vector fields W on Γ, by means of the scalar product

$$(W_1, W_2)_\Gamma = \int_0^1 \langle W_1, W_2\rangle \, dt,$$

in which $\langle \cdot, \cdot \rangle$ is the Riemannian metric, one can introduce the structure of a Hilbert space. The differential expression $\mathcal{L}$, considered on vector fields $W(t)$ of class C^2 satisfying the conditions $W(0) = 0,\ W(1) = 0$, defines in this Hilbert space an operator whose closure L_Γ is a self-adjoint operator. Consider also the operator L^0_Γ, generated by the differential expression

$$L^0W \equiv -\frac{D^2}{dt^2}W.$$

It turns out that the operator $L_\Gamma L_\Gamma^{0-1}$ differs from the identity by a nuclear summand. Therefore the determinant $\det L_\Gamma L_\Gamma^{0-1}$ exists.

Suppose that the geodesic Γ is the image of the rectilinear segment $\gamma = tv$ ($t \in [0,1]$, $v \in TM_p$, $p = \Gamma(0)$ is the initial point of the geodesic) under the exponential mapping $\exp_p$ of the tangent space TM_p into M. Denote by $d\exp|_\Gamma$ the differential of the exponential mapping at the point $v \in TM_p$. One may regard $d\exp|_\Gamma$ as given on the space TM_p, while its values belong to TM_q ($q = \Gamma(1)$ is the endpoint of the geodesic Γ). Let P_Γ be the operator of parallel transport along Γ. The determinant

$$J \equiv \det\bigl(d\exp|_\Gamma P_\Gamma^{-1}\bigr)$$

has meaning.

The quantity J admits a simple geometric interpretation. If τ_0 is an element of Riemannian volume on TM_p, then $\tau = J\tau_0$ is the element of Riemannian volume on M, carried over to TM_p by means of the exponential mapping.

In the present note the following is proved.

Theorem. *The following formula holds (the trace formula)*

$$\widehat{\det} L_\Gamma L_\Gamma^{0^{-1}} = \det\bigl(d\exp|_\Gamma P_n^{-1}\bigr).$$

Below we give a proof of this theorem. It is necessary to note that the trace formula can be completely described in terms of the geodesic flow. In this case the assumption that the flow is geodesic is not necessary. Some generalizations of similar formulas in other directions are also possible, for example, to partial differential equations. These assertions will not be discussed in more detail here.

The operator L_Γ. Let $P(t)$ be the operator of parallel translation along Γ from the point p to the point $\Gamma(t)$. The differential expressions $\mathcal{L}$ and $\mathcal{L}^0$ may be represented in the form

$$\mathcal{L}W = Plw, \qquad \mathcal{L}^0 W = Pl^0 w,$$

where $w = w(t) = P^{-1}(t)W(t) \in TM_p$ is a vector field on γ, and

$$lw = -\frac{d^2}{dt^2}w + q_{vw}, \qquad l^0 w = -\frac{d^2}{dt^2}w,$$

where q_v is a Hermitian transformation in TM_p, given by the formula

$$q_{vw} = -P^{-1}R(V, Fw)V.$$

The differential expressions lw and $l^0 w$ are the usual Sturm–Liouville differential expressions.

The Hilbert space of vector fields on Γ, introduced above, will be denoted by $L_2(\Gamma)$. Analogously, the Hilbert space of vector fields on γ with scalar product

$$(w_1, w_2)_\gamma = \int_0^1 \langle w_1, \overline{w_2} \rangle \, dt$$

will be denoted by $L_2(\gamma)$. In this space the differential expressions l and l^0, considered with zero boundary conditions at the points $t = 0, 1$, define essentially self-adjoint operators. Let their closures be L_γ and L_γ^0.

The operator $U : L_2(\gamma) \to L_2(\Gamma)$, acting by the formula $(Uu)(t) = P(t)u(t)$, maps $L_2(\gamma)$ onto $L_2(\Gamma)$ isometrically. From the relation described above between $\mathcal{L}, \mathcal{L}^0$ and l, l^0 it follows that

Lemma 1. *The differential expressions $\mathcal{L}$ and $\mathcal{L}^0$, considered on vector fields $W(t)$ satisfying the conditions $W(0) = 0$, $W(1) = 0$, define essentially self-adjoint operators, whose closures L_Γ and L_Γ^0 are related to L_γ and L_γ^0 by the formulas $L_\Gamma = UL_\gamma U^{-1}$, $L_\Gamma^0 = UL_\gamma^0 U^{-1}$.*

The resolvent of the operator L_Γ. Let $\Phi_\lambda(t, s)$ (λ a complex number, $0 \leq t, s \leq 1$) be the solution of the equation $(\mathcal{L}_1 - \lambda)\Phi_\lambda(t, s) = 0$, satisfying the conditions: 1) $\Phi_\lambda(s, s) = 0$, 2)

$$\frac{D}{dt} \Phi_\lambda(t, s) \Big|_{t=s} = I(s),$$

where $I(s)$ is the identity transformation of $TM_{\Gamma(s)}$; $\Phi_\lambda(t, s)$ is a linear mapping of $TM_{\Gamma(s)}$ into $TM_{\Gamma(t)}$.

Let ψ be a linear mapping of $TM_{\Gamma(s)}$ into $TM_{\Gamma(t)}$. Define $\psi^T : TM_{\Gamma(t)} \to TM_{\Gamma(s)}$ by the formula

$$\langle \psi W_1, W_2 \rangle_{\Gamma(t)} = \langle W_1, \psi_2^{TW} \rangle_{\Gamma(s)}.$$

Introduce the resolvent $R_\lambda = (L_\Gamma - \lambda I)^{-1}$ of the operator L_Γ. Repeating the classical constructions pertaining to the Sturm–Liouville problem, we obtain the following assertions.

Lemma 2. *The resolvent R_λ outside the spectrum of L_Γ is an integral operator*

$$(R_\lambda W)(t) = \int_0^1 ds \, R_\lambda(t, s) W(s)$$

with continuous kernel $R_\lambda(t, s) : TM_{\Gamma(s)} \to TM_{\Gamma(t)}$ $(0 \leq s,\ t \leq 1)$. The kernel satisfies the relation $R_\lambda(t, s) = R_\lambda^T(s, t)$ and for $t \leq s$ admits the representation

$$R_\lambda(t,s) = \Phi_\lambda(t,0)\Lambda^{-1}(\lambda)\Phi_\lambda^T(s,1),$$

in which

$$\Lambda(\lambda) = \Phi_\lambda^T(t,1)\left(\frac{D}{dt}\Phi_\lambda(t,0)\right) - \left(\frac{D}{dt}\Phi_\lambda^T(t,1)\right)\Phi_\lambda(t,0).$$

$\Lambda(\lambda)$ does not depend on t.

It is known that the spectrum of the operator L_γ, and hence also of the operator L_Γ, consists of eigenvalues λ_n, which have the single limiting point $+\infty$, and n^2/λ_n is bounded as $n \to \infty$. Therefore R_λ, if $\lambda \neq \lambda_{n_1}$, is a nuclear operator, and there exists a trace $\widehat{\mathrm{Sp}} R_\lambda$, which can be computed by the formula

$$\widehat{\mathrm{Sp}} R_\lambda = \int_0^1 dt\ \mathrm{Sp}\, R_\lambda(t,t),$$

where $\mathrm{Sp}\, R_\lambda(t,t)$ is the trace of the transformation $R_\lambda(t,t) : TM_{\Gamma(t)} \to TM_{\Gamma(t)}$. Using Lemma 2 and the formula

$$\Phi_\lambda^T(t,0)\dot{\Phi}_\lambda(t,1) = -\frac{D}{dt}\left[\Phi_\lambda^T(t,0)\left(\frac{D}{dt}\dot{\Phi}_\lambda(t,1)\right) - \left(\frac{D}{dt}\Phi_\lambda^T(t,0)\right)\dot{\Phi}_\lambda(t,1)\right],$$

in which the dot denotes differentiation with respect to λ, we obtain the following result.

Lemma 3. For $\lambda \neq \lambda_n$, R_λ is a nuclear operator and

$$\widehat{\mathrm{Sp}} R_\lambda = -\,\mathrm{Sp}\,\Lambda^{-1}(\lambda)\dot{\Phi}_\lambda(1,0) = -\,\mathrm{Sp}\,\Phi_\lambda^{-1}(1,0)\dot{\Phi}_\lambda(1,0) = -\frac{d}{d\lambda}\ln\det\Phi_\lambda(1,0)P_\Gamma^{-1}.$$

All assertions of the present subsection are applicable also to the resolvent $R_\lambda^0 = (L_\Gamma^0 - \lambda I)^{-1}$, with the replacement of the solution $\Phi_\lambda(t,s)$ of the equation $(\mathcal{L} - \lambda)\Phi = 0$ by the solution $\Phi_\lambda^0(t,s)$ of the equation $(\mathcal{L}^0 - \lambda)\Phi^0 = 0$. It is easy to see that

$$\Phi_\lambda^0(t,s) = \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}}\,P(t)P^{-1}(s).$$

Proof of the theorem. For $\lambda \neq n^2\pi^2$ $(n = 1, 2, ...)$ the function

$$\mathfrak{D}(\lambda) = \widehat{\det}(L_\Gamma - \lambda I)(L_\Gamma^0 - \lambda I)^{-1} = \widehat{\det}\left[I + Q_\nu R_\lambda^0\right]$$

is defined and holomorphic. It is known (^{2}) that

$$\frac{\dot{\mathfrak{D}}(\lambda)}{\mathfrak{D}(\lambda)} = \widehat{\mathrm{Sp}} R_\lambda^0 - \widehat{\mathrm{Sp}} R_\lambda.$$

Using Lemma 3, we obtain

$$\frac{\dot{\mathfrak{D}}(\lambda)}{\mathfrak{D}(\lambda)} = \frac{d}{d\lambda} \ln \det \Phi_\lambda(1,0) \Phi_\lambda^{0-1}(1,0).$$

The nuclear norm of the operator R_λ^0 tends to zero as $\lambda \to -\infty$, therefore

$$\lim_{\lambda \to -\infty} \mathfrak{D}(\lambda) = 1.$$

Further, for fixed t and s,

$$\Phi_\lambda(t,s) \Phi_\lambda^{0-1}(t,s) \xrightarrow[\lambda \to -\infty]{} I(t).$$

Hence it follows that

Lemma 4. The formula holds

$$\mathfrak{D}(\lambda) = \det \Phi_\lambda(1,0) \Phi_\lambda^{0-1}(1,0).$$

Finally, the following holds.

Lemma 5. The relation is valid

$$\Phi_0(t,0) = t\, d \exp_p.$$

(The exponential mapping on the right-hand side acts from TM_p to $TM_{\Gamma(t)}$.)

Proof of the lemma. Consider the geodesic variation

$$\alpha(t,\tau) = \exp_p t(v + \tau u), \qquad \tau \in (-\varepsilon, \varepsilon), \quad \varepsilon > 0, \quad u \in TM_p$$

of the geodesic Γ. The variation vector field

$$W : t \mapsto \frac{\partial}{\partial \tau} \alpha(t,\tau)\Big|_{\tau=0},$$

as is known, satisfies the Jacobi equation $\mathcal{L}W = 0$. It is obvious that $W(t) = (t d \exp_p) u$. From the properties of the exponential mapping it follows that

$$W(0) = 0, \qquad \frac{D}{dt} W \Big|_{t=0} = u.$$

The assertion of the lemma follows from this.

Substituting the formula of Lemma 5 into the formula of Lemma 4 and taking into account the explicit form of $\Phi_\lambda^0(t, s)$ given above, we obtain

$$\mathcal{D}'_\lambda(0) = \widehat{\det}\, L_\Gamma L_\Gamma^{0^{-1}} = \det(d \exp|_\Gamma P_\Gamma^{-1}).$$

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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