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THEORY OF ELASTICITY

1968

SovietRxiv

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Abstract

Full Text

UDC 5.53.539.3

THEORY OF ELASTICITY

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STRESSES AND DEFORMATIONS IN A MULTILAYER HALF-SPACE UNDER A STRIP LOAD ON THE SURFACE

(Presented by Academician L. I. Sedov, 3 VI 1968)

By analogy with the axisymmetric problem ([1,2]), the stresses in a multilayer half-space under plane deformation are expressed through the biharmonic function F in the following form:

$$\begin{aligned} \sigma_y &= \frac{\partial}{\partial y}\left((2-\mu)\nabla^2 F - \frac{\partial^2 F}{\partial y^2}\right), \\ \sigma_x &= -\frac{\partial}{\partial y}\left((1-\mu)\nabla^2 F - \frac{\partial^2 F}{\partial y^2}\right), \\ \tau_{xy} &= \frac{\partial}{\partial x}\left((1-\mu)\nabla^2 F - \frac{\partial^2 F}{\partial y^2}\right). \end{aligned} \tag{1}$$

According to the well-known formulas of elasticity theory ([1]), for the deformations we obtain

$$\begin{aligned} u &= -\frac{1+\mu}{E}\frac{\partial^2 F}{\partial x\,\partial y}, \\ v &= \frac{1+\mu}{E}\left(2(1-\mu)\nabla^2 F - \frac{\partial^2 F}{\partial y^2}\right). \end{aligned} \tag{2}$$

Here μ is Poisson' s ratio; E is the modulus of elasticity; ρ is the density of the layer material; u is the horizontal and v the vertical deformation* (see Fig. 1).

Fig. 1. Diagram of a multilayer half-space under the action of a normal load $p(x)$ and a tangential load $t(x)$

Following (2), the expressions for F may be taken in the following form:

a) for the n-th layer, extending to infinity:

$$F_n = \frac{2H}{a\pi}\int_0^\infty \{A + B[\beta(x-1) + 2\mu_n]\}e^{-\beta x}\frac{\cos\beta\xi}{\beta}\,d\beta,$$

* In the terminology of elasticity theory, u and v are called displacements. We shall call them deformations, as is customary in soil mechanics.

b) for the i-th layer:

$$F_i = \frac{2H}{a\pi}\int_0^\infty \{A + B[\beta(\varkappa-1) + 2\mu_i]\}e^{-\beta\varkappa}\frac{\cos\beta\xi}{\beta}\,d\beta +$$

$$+\frac{2H}{a\pi}\sum_{k=i}^{n-1}\int_0^\infty \{C_k[(1-2\mu_i)\operatorname{sh}N\beta + N\beta\operatorname{ch}N\beta] +$$

$$+D_k[2\mu_i\operatorname{ch}N\beta - N\beta]\}\frac{\cos\xi\beta}{\beta}\,d\beta.$$

Here $\xi = x/H;\ \varkappa = y/H;\ N = r_{k+1} - \varkappa;\ r_i = h_i/H;\ A,\ B,\ C_k,\ D_k$ are parameters that are functions of β and are determined from the boundary conditions.

Substituting the values of F into (1) and (2), we obtain

$$\sigma_{yi} = \frac{2}{a\pi H^2}\int_0^\infty \{A + B[1 + \beta(\varkappa-1)]\}e^{-\beta\varkappa}\beta^2\cos\xi\beta\,d\beta +$$

$$+\frac{2}{a\pi H^2}\sum_{k=i}^{n-1}\int_0^\infty \{C_{kN}\beta\operatorname{sh}N\beta + D_k[\operatorname{sh}N\beta - N\beta\operatorname{ch}N\beta]\}\beta^2\cos\xi\beta\,d\beta;$$

$$\sigma_{xi} = -\frac{2}{a\pi H^2}\int_0^\infty \{A - B[1 - \beta(\varkappa-1)]\}e^{-\beta\varkappa}\beta^2\cos\xi\beta\,d\beta -$$

$$-\frac{2}{a\pi H^2}\sum_{k=i}^{n-1}\int_0^\infty \{C_k[2\operatorname{ch}N\beta + N\beta\operatorname{sh}N\beta] -$$

$$-D_k[\operatorname{sh}N\beta + N\beta\operatorname{ch}N\beta]\}\beta^2\cos\xi\beta\,d\beta;$$

$$\tau_{xyi} = \frac{2}{a\pi H^2}\int_0^\infty [A + \beta B(\varkappa - 1)]e^{-\beta\varkappa}\beta^2 \sin\xi\beta\, d\beta +$$

$$+\frac{2}{a\pi H^2}\sum_{k=i}^{n-1}\int_0^\infty \{C_k[\operatorname{sh} N\beta + N\beta \operatorname{ch} N\beta] - D_{kN}\beta \operatorname{sh} N\beta\}\beta^2 \sin\xi\beta\, d\beta;$$

$$u_i = -\frac{2(1+\mu_i)}{a\pi E_{iH}}\int_0^\infty \{A - B[(1-2\mu_i) - \beta(\varkappa - 1)]\}e^{-\beta\varkappa}\beta \sin\xi\beta\, d\beta -$$

$$-\frac{2(1+\mu_i)}{a\pi E_{iH}}\sum_{k=i}^{n-1}\int_0^\infty \{C_k[2(1-2\mu_i)\operatorname{ch} N\beta + N\beta \operatorname{sh} N\beta] +$$

$$+D_k[(2\mu_i - 1)\operatorname{sh} N\beta - N\beta \operatorname{ch} N\beta]\}\beta \sin\xi\beta\, d\beta;$$

$$v_i = -\frac{2(1+\mu_i)}{a\pi E_{iH}}\int_0^\infty \{A + B[2(1-\mu_i) + \beta(\varkappa - 1)]\}e^{-\beta\varkappa}\beta \cos\xi\beta\, d\beta +$$

$$+\frac{2(1+\mu_i)}{a\pi E_{iH}}\sum_{k=i}^{n-1}\int_0^\infty \{C_k[(1-2\mu_i)\operatorname{sh} N\beta - N\beta \operatorname{ch} N\beta] -$$

$$-D_k[2(1-\mu_i)\operatorname{ch} N\beta - N\beta \operatorname{sh} N\beta]\}\beta \cos\xi\beta\, d\beta.$$

The stresses and deformations in the lower, n-th, layer are determined by the first integrals.

The total number of unknown parameters $A,\ B,\ C_k,\ D_k$ is equal to $4n$; however, with the adopted form of the functions F, the conditions at infinity (as $y \to \infty$, $\sigma_{yn} \to 0,\ \tau_{xyn} \to 0$) and the continuity of stresses ($\sigma_{y\,\text{upper}} = \sigma_{y\,\text{lower}};\ \tau_{xy\,\text{upper}} = \tau_{xy\,\text{lower}}$) at the contact of the layers are satisfied automatically. Thus

Thus, the remaining conditions are used to determine C_k and D_k. With joint displacement of the layers these will be the conditions of continuity of the strains. Using these conditions at the contact of the n-th and $(n-1)$-st layers, we obtain

$$C_{n-1} = \frac{1}{2}(\lambda A - \xi B)e^{-\beta}; \qquad D_{n-1} = \frac{1}{2}[\lambda A + 2(m_{n-1} - 1)B]e^{-\beta}.$$

Here

$$\lambda = \frac{m_{n-1}}{1-\mu_n} - \frac{1}{1-\mu_{n-1}}; \qquad m_{n-1} = \frac{E_{n-1}(1-\mu_n^2)}{E_n(1-\mu_{n-1}^2)};$$

$$\xi = \frac{m_{n-1}(1-2\mu_n)}{1-\mu_n} - \frac{1-2\mu_{n-1}}{1-\mu_{n-1}}.$$

From the equations for u_i and v_i it follows that

$$u_i = u^{\vee}_{i+1} - \frac{4(1-\mu_i^2)}{a\pi E_i H}\int_0^\infty C_i\beta\sin\xi\beta\,d\beta,$$

$$v_i = v^{\vee}_{i+1} - \frac{4(1-\mu_i^2)}{a\pi E_i H}\int_0^\infty D_i\beta\cos\xi\beta\,d\beta,$$

where $u^{\vee}_{i+1}$ and $v^{\vee}_{i+1}$ denote the expressions u_{i+1} and v_{i+1} when the elastic constants of the $(i+1)$-st layer are replaced by those of the i-th layer. Consequently, at the boundaries of the layers the parameters C_i and D_i separate out and can be expressed through the corresponding parameters of the underlying layers. From the conditions $u_i = u_{i+1};\ v_i = v_{i+1}$ at $x = r_{i+1}$ we obtain

$$C_i = \frac{1}{2(1-\mu_i)}\Big\{[A(m_i-1) - B(m_{iV_{i+1,i+1}} - V_{i+1,i})]e^{-\beta r_{i+1}} +$$

$$+\sum_{i+1}^{n-1}[C_k(m_{iW_{i+1,i+1,k}} - W_{i+1,i,k}) + D_k(m_{iQ_{i+1,i+1,k}} - Q_{i+1,i,k})]\Big\};$$

$$D_i = \frac{1}{2(1-\mu_i)}\Big\{[A(m_i-1) + B(m_{iL_{i+1,i+1}} - L_{i+1,i})]e^{-\beta r_{i+1}} +$$

$$+\sum_{i+1}^{n-1}[D_k(m_{iM_{i+1,i+1,k}} - M_{i+1,i,k}) - C_k(m_{iS_{i+1,i+1,k}} - S_{i+1,i,k})]\Big\}.$$

Here

$$m_i = \frac{E_i(1+\mu_{i+1})}{E_{i+1}(1+\mu_i)}; \qquad V_{i+1,j} = 1 - 2\mu_j - \beta(r_{i+1}-1);$$

$$L_{i+1,j} = 2(1-\mu_j) + \beta(r_{i+1}-1); \qquad W_{i+1,j,k} = 2(1-\mu_j)\operatorname{ch}K\beta +$$

$$+K\beta\operatorname{sh}K\beta; \qquad Q_{i+1,j,k} = (2\mu_j - 1)\operatorname{sh}K\beta - K\beta\operatorname{ch}K\beta;$$

$$M_{i+1,j,k} = 2(1-\mu_j)\operatorname{ch}K\beta - K\beta\operatorname{sh}K\beta; \qquad S_{i+1,j,k} = (1-2\mu_j)\operatorname{sh}K\beta - K\beta\operatorname{ch}K\beta: \quad K = r_{k+1} - r_{i+1}.$$

The parameters A and B are easily determined from the conditions on the surface. In the presence of normal $p(x)$ and tangential $t(x)$ loads, these conditions are written as follows:

$$\frac{2}{a\pi H^2}\int_0^\infty \Big\{A + B(1-\beta) + \sum_i^{n-1}[C_k\beta r_{k+1}\operatorname{sh}\beta r_{k+1} +$$

$$+D_k(\operatorname{sh}\beta r_{k+1} - \beta r_{k+1}\operatorname{ch}\beta r_{k+1})]\Big\}\,\beta^2\cos\xi\beta\,d\beta = -p(x);$$

$$\frac{2}{a\pi H^2}\int_0^\infty \Big\{A - \beta B + \sum_i^{n-1}[C_k(\operatorname{sh}\beta r_{k+1} + \beta r_{k+1}\operatorname{ch}\beta r_{k+1}) -$$

$$-D_k\beta r_{k+1}\operatorname{sh}\beta r_{k+1}]\Big\}\,\beta^2\sin\xi\beta\,d\beta = -t(x).$$

In this case $p(x)$ and $t(x)$ are expressed in the form of a Fourier integral.

For a rectangular load of intensity p, for example, this integral after transformations has the form

$$p(x) = \frac{2p}{\pi}\int_0^\infty \sin\left(\frac{a}{H}\beta\right)\cos\left(\frac{x}{H}\beta\right)\frac{d\beta}{\beta}.$$

In the absence of $p(x)$ or $t(x)$, the right-hand side of the corresponding equation is equal to zero.

Thus, the constructed scheme makes it possible, proceeding from bottom to top, successively to determine all unknown parameters related to one another by chain dependence, which considerably simplifies the calculations, since in the end everything reduces to solving a system of two equations with two unknowns A and B of the form

$$f_{11}(\beta)A + f_{12}(\beta)B = F_1(\beta); \qquad f_{21}(\beta)A + f_{22}(\beta)B = F_2(\beta).$$

The corresponding matrix has the form

$$\Phi = \begin{bmatrix} f_{11}(\beta) & f_{12}(\beta) \\ f_{21}(\beta) & f_{22}(\beta) \end{bmatrix}.$$

The solution is carried out on an electronic digital computer, obtaining by one of the existing methods the inverse matrix

$$\Phi^{-1} = \begin{bmatrix} \varphi_{11}(\beta) & \varphi_{12}(\beta) \\ \varphi_{21}(\beta) & \varphi_{22}(\beta) \end{bmatrix}.$$

As a result, A and B are found from the expressions

$$A = \varphi_{11}(\beta)F_1(\beta) + \varphi_{12}(\beta)F_2(\beta); \qquad B = \varphi_{21}(\beta)F_1(\beta) + \varphi_{22}(\beta)F_2(\beta).$$

The values $f_{11}, \dots, f_{22}, F_1(\beta)$ and $F_2(\beta)$ are easily found for specific conditions from the formulas given above. Thus, for example, for the second layer of a three-layer medium under a rectangular normal load on the surface,

$$f_{11}(\beta) = 1 + \frac{1}{2}\lambda e^{-\beta}[(1 + r_2\beta)\,\mathrm{sh}\, r_2\beta - r_2\beta\,\mathrm{ch}\, r_2\beta];$$

$$f_{12}(\beta) = (1-\beta) + e^{-\beta}[(m_2 - 1) - \tfrac{1}{2}\xi r_2\beta]\,\mathrm{sh}\, r_2\beta - e^{-\beta}(m_2 - 1)r_2\beta\,\mathrm{ch}\, r_2\beta;$$

$$F_1(\beta) = apH^2 \sin\frac{a\beta}{H}\Big/\beta^3,$$

where $r_2 = h_2/H$.

The problem considered has practical significance for the calculation of engineering structures made of layers of heterogeneous materials. With a small number of layers it is possible to express the parameters $A, \dots, D$ through E, μ, β and to obtain expressions for stresses and strains directly as functions of the coordinates of the point and of the elastic constants of the layers.

The author expresses his gratitude to Prof. A. F. Smirnov for a number of valuable suggestions.

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Received
12 X 1968

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Note: Figure translations are in progress. See original paper for figures.