ON COMPLETE REGULARITY OF MULTIDIMENSIONAL STATIONARY PROCESSES

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 519.21

MATHEMATICS

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ON COMPLETE REGULARITY OF MULTIDIMENSIONAL STATIONARY PROCESSES

(Presented by Academician Yu. V. Linnik on 18 IV 1968)

The purpose of the present note is to obtain analogues of Theorem 1 from ([1]) and Theorem 4 from ([2]) for the case of multidimensional completely regular stationary processes (cf. ([3], [4])).

1. Let first $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ be a stationary process with discrete time $t = \dots, 1, 0, 1, \dots$. The process $\xi(t)$ is completely regular if

$$\sup |\mathbf{E}x\bar{y}| = \rho(\tau) \longrightarrow 0, \qquad \tau \to \infty,$$

where the supremum is taken over all $x \in H_{-\infty}^{0}$, $y \in H_{\tau}^{\infty}$, $\mathbf{E}|x|^2 = \mathbf{E}|y|^2 = 1$, and H_a^b denotes the linear closed (in the mean-square sense) span of the random variables $\xi_j(t)$, $j = 1, \dots, n$, $a \le t \le b$. Denote by $f(\lambda) = \|f_{ij}(\lambda)\|$ the spectral matrix density of $\xi(t)$.

Theorem 1. *The spectral density $f(\lambda)$ of a completely regular process is always representable in the form*

$$f(\lambda) = \mathbf{P}(e^{i\lambda})\, g(\lambda)\mathbf{P}^*(e^{i\lambda}), \qquad \mathbf{P} = \|P_{ij}\|, \qquad g = \|g_{ij}\|.$$

Here $\mathbf{P}(z)$ is a diagonal polynomial matrix, and the primitives $G_{ij}(\lambda)$ of the functions $g_{ij}(\lambda)$ satisfy the condition

$$\omega_{ij}(\delta) = \sup_{\lambda,\, |h| \le \delta} \frac{|G_{ij}(\lambda + h) + G_{ij}(\lambda - h) - 2G_{ij}(\lambda)|}{|G_{ii}(\lambda + h) - G_{ii}(\lambda)|^{1/2} |G_{jj}(\lambda + h) - G_{jj}(\lambda)|^{1/2}} \longrightarrow 0. \quad (1)$$

We give the scheme of the proof of this theorem. By Theorem 1 from ([1]), for all $i = 1, \dots, n$,

$$f_{ii}(\lambda) = |P_{ii}(e^{i\lambda})|^2 g_{ii}(\lambda),$$

where $P_{ii}(z)$ is a polynomial with roots on $|z| = 1$, and $G_{ii}(\lambda)$ satisfies condition (1). We now define the matrix $\mathbf{P}(z)$ as the diagonal matrix with diagonal elements $P_{ii}(z)$, and then define the matrix $g(\lambda)$ from the equality

$$f(\lambda) = \mathbf{P}(e^{i\lambda}) g(\lambda) \mathbf{P}^*(e^{i\lambda}).$$

Lemma 1. *The matrix $g(\lambda)$ is the spectral matrix density of a stationary completely regular process.*

In proving Lemma 1 it suffices to consider only the case when one of the diagonal elements of the matrix $\mathbf{P}(e^{i\lambda})$ has the form $e^{i\lambda} - e^{i\theta}$, while all the others are equal to one. In this case the proof is carried out in the same way as the proof of Lemma 3.1 from ([1]).

Put

$$h_i(t; \mu) = h_i(t) = \frac{1}{t} \int_{-\pi}^{\pi} \frac{\sin^2(t\lambda/2)}{\sin^2(\lambda/2)} g_{ii}(\lambda + \mu)\, d\lambda.$$

The principal role in the proof of Theorem 1 is played by the following

Lemma 2. *Let $a(\lambda)$ be an odd, three times differentiable function with bounded third derivative, vanishing outside* $[-1, 1]$.

Then, as $t \to \infty$, uniformly in μ,

$$\frac{1}{t} \left| \int_{-\pi}^{\pi} \frac{\sin^2(t\lambda/2)}{\sin^2(\lambda/2)} g_{ij}(\lambda + \mu)\, a(t\lambda)\, d\lambda \right| = o\left(\sqrt{h_i(t) h_j(t)} \right). \tag{2}$$

The proof of the lemma is too cumbersome to present here; we note only that it is based on ideas similar to those used in [1] in the proof of Lemma 3.3.

Let now $\varepsilon > 0$ be an arbitrary positive number. Define an even, three-times differentiable function $a_\varepsilon(\lambda)$ so that on $[\varepsilon, 1-\varepsilon]$ $a_\varepsilon(\lambda)$ coincides with $(\lambda/2)^2 \sin^{-2}(\lambda/2)$, is equal to zero outside $[-1, 1]$, increases monotonically on $[-\varepsilon, \varepsilon]$, and decreases monotonically on $[1-\varepsilon, 1]$. Substituting in (2) a_ε in place of a, using the properties of the functions $h_i(t)$ proved in [1], and arguing in the same way as at the end of the proof of Theorem 1 of [1], we arrive at (1), in any case, if $i \neq j$. In the case $i = j$, equality (1) follows from [1].

From Theorem 1 there immediately follows the following result, which is a strengthening of Theorem 4 of [4].

The functions $f_{ij}(\lambda)$ have no discontinuities of the first kind at those points λ_0 where

$$\int_{\lambda_0-\delta}^{\lambda_0+\delta} f_{ii}(\lambda)\,d\lambda \cdot \int_{\lambda_0-\delta}^{\lambda_0+\delta} f_{jj}(\lambda)\,d\lambda = O(\delta^2), \qquad \delta \to 0.$$

Theorem 2 (cf. [4]). Let the spectral density $\mathbf{f}(\lambda)$ of the process $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ be representable in the form

$$\mathbf{f}(\lambda) = \mathbf{P}(e^{i\lambda})\mathbf{g}(\lambda)\mathbf{P}^*(e^{i\lambda}),$$

where $\mathbf{P}(z)$ is a polynomial matrix, all elements of the matrix $\mathbf{g}(\lambda)$ are bounded and satisfy condition (1), and moreover

$$\sum_k w_{ij}^2(2^{-k}) < \infty, \qquad 0 < m \le \det \mathbf{g}(\lambda), \qquad 0 < m \le g_{ii}(\lambda), \quad i = 1, \dots, n.$$

Then the process $\xi(t)$ is completely regular.

2. Let now $\xi(t)$ be a process with continuous time t, $-\infty < t < \infty$. We shall still denote the spectral (matrix) density of the process $\xi(t)$ by $\mathbf{f}(\lambda)$.

Theorem 3. Whatever positive number a may be, the spectral density $\mathbf{f}(\lambda)$ of a completely regular process $\xi(t)$ can be written in the form

$$\mathbf{f}(\lambda) = \mathbf{P}_a(\lambda)\mathbf{g}(\lambda)\mathbf{P}_a^*(\lambda),$$

where $\mathbf{P}(\lambda)$ is a diagonal polynomial matrix, and the transformed $G_{ij}(\lambda)$ of the functions $g_{ij}(\lambda)$, $g_a = \|g_{ij}\|$, satisfy the condition

$$\sup_{|\lambda|\le a,\ |h|\le\delta} \frac{\left|G_{ij}(\lambda+h) + G_{ij}(\lambda-h) - 2G_{ij}(\lambda)\right|}{\left|G_{ii}(\lambda+h) - G_{ii}(\lambda)\right|^{1/2}\left|G_{jj}(\lambda+h) - G_{jj}(\lambda)\right|^{1/2}} = \omega_{ij}^{(a)}(\delta) \to 0. \quad (3)$$

Theorem 4 (cf. [4]). Let the spectral density $\mathbf{f}(\lambda)$ of the process $\xi(t)$ be representable in the form

$$\mathbf{f}(\lambda) = \mathbf{B}(\lambda)\mathbf{g}(\lambda)\mathbf{B}^*(\lambda),$$

where all elements $B_{ij}(\lambda)$ of the matrix $\mathbf{B}(\lambda)$ are entire functions of finite degree, square-summable. Suppose further that all elements of the matrix $\mathbf{g}$ are bounded, satisfy condition (3) with $a = \infty$, and moreover

$$\sum_k \omega_{ij}^2(2^{-k}) < \infty, \qquad 0 < m \le \det \mathbf{g}(\lambda), \qquad 0 < m \le g_{ii}(\lambda).$$

Then the process $\xi(t)$ is completely regular.

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Received
14 III 1968

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.