s-CAPACITY AND THE BEHAVIOR OF THE SOLUTION OF A SECOND-ORDER ELLIPTIC EQUATION WITH DISCONTINUOUS COEFFICIENTS IN A NEIGHBORHOOD OF A BOUNDARY POINT

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s-CAPACITY AND THE BEHAVIOR OF THE SOLUTION OF A SECOND-ORDER ELLIPTIC EQUATION WITH DISCONTINUOUS COEFFICIENTS IN A NEIGHBORHOOD OF A BOUNDARY POINT

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In this note we consider a linear elliptic equation of the second order

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0. \tag{1}$$

It is assumed that there exist two positive constants α and M such that, in the domain where the equation is considered,

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \geq \alpha|\xi|^2, \qquad \sum_{i=1}^{n} a_{ii} < M; \tag{2}$$

$$|b_i| < 1, \qquad -1 < c \leq 0. \tag{3}$$

By a solution of such an equation we shall mean a classical solution—a continuous function possessing the derivatives entering into the equation and satisfying the equation identically.

In the note the question of the regularity of a boundary point for the Dirichlet problem for this equation will be considered.

If in equation (1) the coefficients of the highest derivatives are smooth or even merely continuous, but with a modulus of continuity satisfying the Dini condition, then the conditions for regularity of a boundary point are the same as for the Laplace equation ([1−3]). The same will be true if the equation has divergence form (the coefficients in this case may be arbitrary bounded ones) ([4]). In the general case, however, as is shown by the example given in Sec. 3°, this is no longer so.

Definition 1. Let positive numbers α and M be given. A point x^0 of the boundary Γ of a domain D is called α, M-regular with respect to the Dirichlet problem if there exists a continuous function $\omega(r)$, $0 \leq r < r_0$, $\omega(0) = 0$, $\omega(r) > 0$ for $r > 0$, such that, whatever the neighborhood σ of the point x^0, the domain $D' \subset D$ with boundary Γ', and the equation (1), for which inequalities (2) hold with the given constants α, M, and whatever the solution $u(x) < 1$ of this equation, continuous on $D' \cup \Gamma'$ and nonpositive on the intersection $\Gamma' \cap \sigma$, there is a neighborhood σ' of the point x^0, depending only on σ, α, and M, such that $u(x) < \omega(|x - x^0|)$ for $x \in \sigma' \cap D$.

We shall give sufficient conditions for regularity of a boundary point.

1°. ***s*-capacity.** Let s be a positive number. Let E be a B-set of n-dimensional Euclidean space. Consider all possible measures μ, defined on subsets of the set E, such that

$$\int_E \frac{d\mu(y)}{|x-y|^s} \leq 1 \quad \text{for } x \notin E; \tag{4}$$

the number $C_s(E) = \sup \mu(E)$, where the least upper bound is taken over all measures μ satisfying inequality (4), will be called the s-capacity of the set E.

Lemma 1. The s-capacity of a ball of radius R is not less than R^s.

Lemma 2. Let $s > 1$ and let $\Pi_{\rho,h}$ be the cylinder:

$$\sum_{i=1}^{n-1} x_i^2 < \rho^2, \qquad 0 < x_n < h,$$

where $\rho < h$. Then $C_s(\Pi_{\rho,h}) \geq Ch\rho^{s-1}$, where C is a positive constant depending on s.

Lemma 3. Let $s < n$; let E be located inside the ball of unit radius. Then $C_s(E) \geq \eta \operatorname{mes} E$, where $\eta > 0$ is a constant depending on s and n (by $\operatorname{mes} E$ is denoted the Lebesgue measure of the set E).

Lemma 4. For any $a > 0$ and $M > 0$ there exist $s > 0$ and $r_0 > 0$ such that, for every operator L of the form (1) satisfying inequalities (2) and (3), the inequality

$$L\left(1/|x - x^0|^s\right) \geq 0$$

holds, where x^0 is a fixed point of R_n, x is a point of the domain where the operator L is defined, and $|x - x^0| < r_0$.

In this case we shall say that the s-capacity is **upper** for the operator L in radius r_0.

Remark 1. The number s can be made to depend only on the ratio M/a.

The proof of these lemmas is elementary.

2°. Sufficient conditions for regularity of a boundary point. In what follows, by $Q_R^{x^0}$ we shall denote the ball of radius R with center at the point x^0.

Lemma 5 (main). Let a domain D be situated in the ball $Q_{4R}^{x^0}$, have limit points on the surface of the ball $Q_R^{x^0}$, and intersect the ball $Q_R^{x^0}$. Let H be the intersection of the complement of the domain D with the ball $Q_R^{x^0}$. Let Γ be that part of the boundary of the domain D which is situated strictly inside $Q_{4R}^{x^0}$. Let equation (1) be defined in D. Let $u(x)$ be its solution, positive in D, continuous in $\overline{D}$, and vanishing on Γ. Let the numbers $s > 0$ and $r_0 > 0$ be such that the s-capacity is upper for the operator L in radius r_0 and $8R < r_0$.

Then

$$\max_{\overline{D}} u \geq \left[1 + \frac{\xi}{R^s} C_s(H)\right] \max_{D \cap Q_R^{x^0}} u,$$

where $\xi > 0$ is a constant depending on s.

Proof. Choose an arbitrary $\varepsilon > 0$, and let the measure μ be such that

$$U(x) = \int_H \frac{d\mu(y)}{|x-y|^s} \leq 1 \quad \text{for } x \notin H, \qquad \mu(H) > C_s(H) - \varepsilon.$$

Let

$$M = \max_{\overline{D}} u.$$

Put

$$v(x) = M\left[1 - U + \frac{C_s(H)}{(3R)^s}\right].$$

We have $Lv \leq 0$ in D, and v is not less than u on the boundary of the domain D. Therefore

$$\max_{D \cap Q_R^{x^0}} u \leq \max_{D \cap Q_R^{x^0}} v \leq M\left[1 - \frac{C_s(H)}{R^s}\left(\frac{1}{2^s} - \frac{1}{3^s}\right) + \frac{\varepsilon}{(3R)^s}\right],$$

which, by the arbitrariness of ε, proves the lemma.

With the help of Lemma 5 one proves

Lemma 6. Let, in the spherical layer

$$R < |x - x^0| < 4R,$$

there be situated a domain D, having limit points on both spheres $|x - x^0| = R$ and $|x - x^0| = 4R$. Let Γ be that part of the boundary of the domain D which is situated strictly inside the layer. Let equation (1) be defined in D, and let $u(x)$ be its solution, positive in D, continuous in $\overline{D}$, and vanishing on Γ. Let the numbers $s > 0$ and $r_0 > 0$ be such that the s-capacity is upper for the operator L in radius r_0 and $8R < r_0$.

Then

$$\max_{x\in\overline{D}} u(x) > \left[1+\frac{\zeta C_s(H)}{R^s}\right]\max_{x\in D,\ |x-x^0|=2R} u(x),$$

where ζ is a constant depending on s and r_0, and
$H = (Q^{x^0}_{3R} \setminus Q^{x^0}_{2R}) \setminus D$.

With the aid of Lemma 5 one proves

Theorem 1. Let there be a domain D with boundary Γ, and let $x^0 \in \Gamma$. Denote by H the complement of the domain D. Let $\alpha > 0$ and $M > 0$ be given numbers, and let $s > 0$ be such that s-capacity is an upper one for every operator L satisfying inequality (2).

Set $\gamma_m = C_s(H \cap Q^{x^0}_{2^{-m}})$, and suppose that

$$\sum_{m=1}^{\infty} 2^{ms}\gamma_m = \infty. \tag{5}$$

Then the point x^0 is α, M-regular, and, if we denote

$$S(t) = \sum_{m=1}^{[t]} 2^{ms}\gamma_m,$$

then for the function $\omega(r)$, for sufficiently small r, the estimate

$$\omega(r) < e^{-aS(|\ln r|)}$$

will hold, where the constant $a > 0$ depends on α and M.

In the case where the coefficients of the equation satisfy the Hölder condition, an analogous result was obtained in [5].

With the aid of Lemma 6 one proves

Theorem 2. Let D, Γ, and x^0 have the same meaning as in Theorem 1. Let equation (1) be defined in D, and let $s > 0$ be such that s-capacity is upper for the operator L. Let γ_m have the same meaning as above, and let condition (5) be fulfilled. Then a bounded solution of equation (1), continuous on Γ in a neighborhood of x^0, is also continuous at x^0.

Such a point will be called O-regular.

From Lemma 1 and Theorems 1 and 2 it follows that

Theorem 3. If a boundary point of a domain can be touched by a vertex cone from outside the domain, then such a point is regular (in both senses) for any equation (1), and as $\omega(r)$ one may take r^β for sufficiently small $\beta > 0$ (β depends on α and M, more precisely on their ratio).

Lemma 2 makes it possible to give a milder sufficient condition for regularity:

Theorem 4. Let the point x^0 of the boundary Γ of the domain D possess the following property. One can introduce an orthogonal coordinate system $y_1, \dots, y_n$ with origin at this point such that the points belonging to the cusp

$$\left(\sum_{i=1}^{n-1} y_i^2\right)^{1/2} < \frac{y_n}{(\ln 1/y_n)^{1/(s-1)}}, \qquad 0 < y_n < h \tag{6}$$

for some h, $0 < h < 1$, do not belong to the domain D. Then the point x^0 is O-regular for every equation (1) for which s-capacity is upper. For any α and M to which this s corresponds, the point x^0 is α, M-regular.

This condition corresponds to the Uryson condition [6] for the Laplace equation when $n \geq 4$. It is all the more interesting that when $n = 3$, for a general equation (1), in contrast to the Laplace equation, this condition cannot be substantially weakened, as is shown by the example given in the following section.

If $M/\alpha < n + 2$, then one can find $s < n$ for which s-capacity is upper for the corresponding equations. Lemma 3 permits, in this case, replacing in Theorems 1 and 2 the capacity of the intersection $H \cap Q_{2^{-m}}^{x^0}$ by its Lebesgue measure.

3°. One example of an irregular point. For any $\delta > 0$ one can find a homogeneous elliptic equation with three independent

variables

$$\sum_{i,k=1}^{3} a_{ik}(x_1, x_2, x_3) \frac{\partial^2 u}{\partial x_i \partial x_k} = 0,$$

whose coefficients differ from the coefficients of Laplace' s equation by less than δ, defined in a domain D, the boundary Γ of which has a point x^0 of the same type as in Theorem 4, but with another exponent of the logarithm in inequality (6):

$$\left(\sum_{i=1}^{n-1} y_i^2\right)^{1/2} < \frac{y_n}{(\ln 1/y_n)^a},$$

where $a > 0$ is a number depending on δ, and find a solution $u(x)$ of this equation in the domain D such that: $u(x)$ is bounded, $u(x)$ is continuous at all boundary points except x^0, in a neighborhood of x^0 it assumes on the boundary the value equal to one, and, at the same time,

$$\lim_{x \to x^0} u(x) < 1.$$

Let $\varepsilon > 0$ be arbitrary and $a > (1+\varepsilon)/\varepsilon \ln 2$. Put $s = 1+\varepsilon$. Put

$$u(x_1, x_2, x_3) = \sum_{m=N}^{\infty} \frac{2^{-m(s-1)}}{m^s} \int_{2^{-(m+1)}}^{2^{-m}} \frac{dt}{\left[x_1^2 + x_2^2 + (x_3 - t)^2\right]^{s/2}}.$$

The function u is defined outside the positive half-axis x_3. Choose the number N so that $u(0,0,0) < 1/2$, which is possible, since at this point the series converges. The set of points where $u(x_1, x_2, x_3) > 1$ forms a funnel B around the positive half-axis x_3, containing the funnel

$$\left(\sum_{i=1}^{2} x_i^2 \right)^{1/2} < \frac{x_3}{(\ln 1/x_3)^a}, \qquad 0 < x_3 < 2^{-N}.$$

Take as the domain D the ball of radius 2^{-N} with center at the origin, from which the funnel $\bar{B}$ has been removed.

We now construct an equation whose solution is $u(x)$. Through every point $x \in D$ draw a plane π through this point and the axis x_3 (at points of the axis x_3, $x_3 < 0$, one may take as π any plane passing through the axis x_3). Introduce a local orthogonal coordinate system ξ_1, ξ_2, ξ_3 so that the axes ξ_1, ξ_2 lie in the plane π. At this point x, in the coordinates ξ, define the equation as follows:

$$\frac{2}{2+\varepsilon} \left(\frac{\partial^2 u}{\partial \xi_1^2} + \frac{\partial^2 u}{\partial \xi_2^2} \right) + \frac{\partial^2 u}{\partial \xi_3^2} = 0.$$

Remark 2. The condition of nonnegativity of the coefficients $C(x)$ can be replaced by the condition that this coefficient be bounded from above.

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Note: Figure translations are in progress. See original paper for figures.