ON A BOUNDARY-VALUE PROBLEM, ARISING IN APPLICATIONS, FOR A SECOND-ORDER DIFFERENTIAL EQUATION ON THE HALF-LINE

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Abstract

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MATHEMATICS

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ON A BOUNDARY-VALUE PROBLEM, ARISING IN APPLICATIONS, FOR A SECOND-ORDER DIFFERENTIAL EQUATION ON THE HALF-LINE

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In mechanics the following boundary-value problem is known:

$$x''' + 2xx'' + 2\beta(1 - {x'}^2) = 0, \qquad x(0) = x'(0) = 0, \qquad x'(+\infty) = 1, \quad (1')$$

arising in the study of self-similar solutions of boundary-layer equations ([1]). The change of variables $x'(t) = 1 - y(x)$ brings this problem to the form

$$y'' = F(x, y, y'), \qquad y(0) = a, \qquad y(+\infty) = 0. \quad (1)$$

In the investigation of the question of mixing of two gas jets, the boundary-value problem ([2]) was obtained:

$$x''' + xx'' = 0, \qquad x'(-\infty) = a, \qquad x(0) = 0, \qquad x'(+\infty) = 0, \quad (2')$$

which is similarly reduced to the form

$$y'' = F(x, y, y'), \qquad y(-\infty) = a, \qquad y(+\infty) = 0. \quad (2)$$

From physical considerations one can often indicate inequalities that the desired solution must satisfy. Thus, in problem $(1')$ we have $0 \le x'(t) \le 1$, while in problem $(2')$, $0 \le x'(t) \le a$.

Boundary-value problems of the form (1), (2) in various special cases have been studied by many authors ([3–9]). In the present note an existence theorem will be proved for a solution of problem (1) or (2) for a sufficiently broad class of functions F. In particular, the existence theorems in papers ([3–9]) can be obtained from Theorem 1. We shall show how Theorem 1 can be applied in some new cases.*

Assume that there exist functions $f_+(x)$ and $f_-(x)$, continuous for $x \ge 0$, such that the function $F(x, y, z)$ is defined in the strip

$$\pi: \ \{x \ge 0, \ f_-(x) \le y \le f_+(x), \ |z| < \infty\},$$

is continuous in π with respect to x, and satisfies a Lipschitz condition with respect to y and z. Suppose, moreover, that the inequalities

$$f''_+(x) \le F(x, f_+(x), f'_+(x)), \qquad f''_-(x) \ge F(x, f_-(x), f'_-(x)),$$

$$f'_+(x-0) \ge f'_+(x+0), \qquad f'_-(x-0) \le f'_-(x+0),$$

$$|f_+(x)| + |f_-(x)| \le M,$$

$$f_+(0) \ge a \ge f_-(0), \qquad \lim_{x\to\infty} f_-(x) \le 0 \le \lim_{x\to\infty} f_+(x),$$

$$|F(x, y, z)| \le C(x, y)\varphi(z), \tag{A}$$

are satisfied, where $C(x, y)$ is continuous,

$$\int_0^\infty z[\varphi(z)]^{-1}\, dz = \infty.$$

We shall assume that

* The present work was reported at the World Congress of Mathematicians in Moscow in 1966.

the discontinuities f'_+ and f'_- are isolated, of the first kind, and, where $f'_-(f'_+)$ exists, $f''_+(f''_-)$ exists.

In what follows, for simplicity we shall consider problem (1) under the assumption

$$\lim_{x\to\infty} f_-(x) = 0.$$

The considerations in the general case and for problem (2) are entirely analogous.

Suppose that for every γ $(0 < \gamma \le d = \overline{\lim}_{x\to\infty} f_+(x))$ there is a function $F_\gamma(x, y, z)$, nonincreasing in y, such that $F \ge F_\gamma$ in some strip

$$\pi_\gamma : \quad \{|y-\gamma| \le \varepsilon_\gamma < \gamma,\ (x, y, z) \subset \pi\}.$$

Consider the solutions of the equation $y'' = F_\gamma$ satisfying the conditions

$$y(x_0) = \gamma - \delta, \qquad y'(x_0) = \alpha.$$

The exact lower bound of the α' s for which, from the fulfillment on the interval $x_0 \le t \le x$ of the inequality

$$y(t) \le \gamma + \varepsilon_\gamma, \qquad (t, y, y') \subset \pi,$$

there follows the inequality $y(x) \geq \gamma - \varepsilon_\gamma$, will be denoted by $m(x_0, \gamma, \delta)$. If there are no such α, we put $m(x_0, \gamma, \delta) = +\infty$. The solution of the equation $y'' = F_\gamma$ satisfying the conditions

$$y(x_0) = \gamma - \delta, \qquad y'(x_0) = m(x_0, \gamma, \delta),$$

will be denoted by $M(x, x_0, \gamma, \delta)$.

Just as in [8], it can be shown that, under conditions (A), the problem $y(0) = a$, $y(x_n) = y_n$, where $x_n > 0$, $f_-(x_n) \leq y_n \leq f_+(x_n)$, has a solution $y_n(x)$ satisfying, for $0 \leq x \leq x_n$, the inequality

$$f_-(x) \leq y_n(x) \leq f_+(x),$$

and that there exist subsequences $x_{n_k} \to \infty$, $y_{n_k} \to 0$ such that $y_{n_k}(x) \to Y(x)$, $Y(0) = a$. Obviously,

$$f_-(x) \leq Y(x) \leq f_+(x). \tag{3}$$

Theorem 1. *If conditions (A) are fulfilled and for every* γ $(0 < \gamma \leq d)$ *there is a* δ $(0 < \delta < \varepsilon_\gamma)$ *such that*

$$\int^\infty m(x_0, \gamma, \delta)\, dx_0 = -\infty, \qquad \int^\infty \{|m| + m\}\, dx_0 < +\infty,$$

then

$$\lim_{x \to \infty} Y(x) = 0,$$

i.e. the boundary-value problem (1) has a solution satisfying inequality (3).

Proof. Suppose this is not so and

$$\overline{\lim}_{x \to \infty} Y(x) = \gamma > 0.$$

Then there exist points ξ_i such that $Y(\xi_i) \to \gamma$, $\xi_i \to \infty$. Choose A_0 so large that for $\xi_i > A_0$ the inequalities

$$|Y(\xi_i) - \gamma| < \delta/3, \qquad \int_0^\infty \{|m| + m\}\, dx_0 < \delta/3$$

hold. We shall show that for some ξ_i $(\xi_i \to +\infty)$ the inequality

$$Y'(\xi_i) > m(\xi_i, \gamma, \delta)$$

is valid. There are two possible cases: either $|Y(x) - \gamma| < \delta$ for $x > \xi_i$, or

$$\underline{\lim}_{x\to\infty} Y(x) \leq \gamma - \delta.$$

In the first case the inequality

$$Y'(x) \leq m(x, \gamma, \delta)$$

contradicts the fact that

$$\int^{\infty} m\, dx_0 = -\infty,$$

and in the second it contradicts the fact that

$$\int_A^{\infty} \{|m| + m\}\, dx_0 < \delta/3.$$

It follows from what has been said that

$$y_{n_i}(\xi_i) \geq \gamma - \delta, \qquad y'_{n_i}(\xi_i) \geq m(\xi_i, \gamma, \delta)$$

on some subsequence $n_i \to \infty$. Applying the comparison theorem proved in [10], we would obtain that, if $y_{n_i}(x) \leq \gamma + \delta$ for $x \geq \xi_i$, then

$$y_{n_i}(x_{n_i}) \geq M(x_{n_i}, \xi_i, \gamma, \delta) \geq \gamma - \varepsilon_\gamma > 0,$$

which is impossible. Hence $y_{n_i}(x)$ attains at a point $\omega_i > \xi_i$ a maximum γ_i, and $\gamma_i \geq \gamma + \delta$. Denoting $\gamma_0 = \lim_{i\to\infty} \gamma_i$ and arguing as before, we arrive at a contradiction. The theorem is proved.

Let us give a simple corollary of Theorem 1:

If conditions (A) are fulfilled and

$$\lim f_+(x) = \lim f_-(x),$$

then the boundary-value problem (1) has a solution. If $F \geq F_1$ for $x \geq A$, $(x, y, z) \subset \pi$, and the boundary-value problem

$$y(A) = f_+(A), \qquad y(\infty) = 0$$

for the equation $y'' = F_1(x, y, y')$ has

solution $y_1(x)$ lying in π, then the boundary-value problem (1) also has a solution $Y(x)$, and $f_-(x) \leq Y(x) \leq y_1(x)$ for $x \geq A$.

Thus, in order to prove the existence of a solution of problem (1) for a given equation, it suffices to construct $f_+(x)$ and $f_-(x)$, choose F_γ (if $\lim_{x\to\infty} f_-(x) = 0$), and show that $m(x_0, \gamma, \delta)$ has the required properties. Under the restrictions imposed on F in papers ([3–9]), this is not hard to do. Thus, in problem (1′), after the substitution $x' = 1 - y$, we obtain $f_+(x) \equiv 1$, $f_-(x) \equiv 0$, $F \geq -2x\, dy/dx + 4\beta y$, and for $\beta \geq 0$ one may use the corollary of Theorem 1: $0 < y \leq cx^{-(2\beta+1)} e^{-x^2} \times [1 + o(1/x)]$.

Consider now problem (1) for $F = \psi(x)\varphi(y)$. Suppose that φ and ψ are defined and continuous for $x \geq 0$, $y \geq 0$, that φ satisfies a Lipschitz condition and, moreover,

$$\varphi(0) = 0, \quad \varphi(y) > 0 \text{ for } y > 0, \quad \int_0^\infty x\psi(x)\, dx = +\infty, \quad \int_0^\infty x[|\psi| - \psi]\, dx < \infty. \tag{4}$$

If $\psi \geq 0$, this problem was solved in paper ([5]).

Under these assumptions there exists a solution of the equation $y'' = \varphi(y)\psi(x)$ tending to zero and positive for large x. If $\varphi(y)$ has a finite limit as $y \to +\infty$, then in this case the existence of a solution of the boundary-value problem (1) follows from conditions (4). If

$$\varphi(y) \equiv y, \qquad \psi_+ = \frac{1}{2}\{\psi + |\psi|\}, \qquad \psi_- = \frac{1}{2}\{\psi - |\psi|\},$$

$$\int_\xi^x dt \int_\xi^t \psi_-(s)\, ds + q \int_\xi^x dt \int_\xi^t \psi_+(s)\, ds \geq -1 + q \tag{5}$$

for some q $(0 < q < 1)$, all ξ and x $(x > \xi)$, and $\psi(x)$ satisfies conditions (4), then the boundary-value problem (1) has a positive solution. We note that condition (5) is required only for the construction of $f_+(x)$, and therefore it may be replaced by the requirement that a bounded positive solution exist.

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Note: Figure translations are in progress. See original paper for figures.