SOME IDENTITIES FOR FUNCTIONS OF MANY VARIABLES AND THEIR APPLICATION

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Abstract

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SOME IDENTITIES FOR FUNCTIONS OF MANY VARIABLES AND THEIR APPLICATION

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Let the function $f(x_1, \dots, x_n)$ be given at every point of some domain G of the n-dimensional Euclidean space R_n. We introduce the difference forms

$$\Delta^{k}_{u_1,\dots,u_n} f(x_1, \dots, x_n) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x_1 + \nu u_1, \dots, x_n + \nu u_n); \quad (1)$$

$$\Delta^{r_1,\dots,r_n}_{u_1,\dots,u_n} f(x_1, \dots, x_n) = \Delta^{r_1}_{u_1} \cdots \Delta^{r_n}_{u_n} f(x_1, \dots, x_n), \quad (2)$$

where

$$\Delta^{r_i}_{u_i} f(x_1, \dots, x_n) = \sum_{\nu=0}^{r_i} (-1)^{r_i-\nu} \binom{r_i}{\nu} f(x_1, \dots, x_{i-1}, x_i + \nu u_i, x_{i+1}, \dots, x_n). \quad (3)$$

Theorem 1. For any natural k and arbitrary integer $l \geq 0$, the following identities hold, establishing a relation between the total difference (1) and the mixed differences (2):

$$\sum_{\nu_{n-1}=0}^{k} \sum_{\nu_{n-2}=0}^{\nu_{n-1}} \cdots \sum_{\nu_1=0}^{\nu_2} (-1)^{\nu_1+\dots+\nu_{n-1}} 2^{(k-\nu_1)l} k! \, \frac{\Delta^{\nu_1, \nu_2-\nu_1, \dots, \nu_{n-1}-\nu_{n-2}, k-\nu_{n-1}}_{u_1,\dots,u_n} f(x_1, \dots, x_n)}{(k-\nu_{n-1})!(\nu_{n-1}-\nu_{n-2})! \cdots (\nu_2-\nu_1)!\nu_1!} =$$

$$= 2^{kl} \Delta^{k}_{-u_1/2^l, -u_2, \dots, -u_{n-1}, u_n} f(x_1, \dots, x_{n-2}, x_{n-1} + k u_{n-1}, x_n) + A_{k+1,l}(f), \quad (4)$$

where $A_{k+1,0}(f) = 0$, and for $l \geq 1$

$$A_{k+1,l}(f)=\sum_{\nu_{n-1}=0}^{k}\sum_{\nu_{n-2}=0}^{\nu_{n-1}}\cdots\sum_{\nu_1=0}^{\nu_2}(-1)^{\nu_1+\cdots+\nu_{n-1}}\times$$

$$\times\sum_{\mu=0}^{\nu_1}\sum_{p=0}^{l-1}\frac{2^{(k-\nu_1)l+p\nu_1}k!\,S_{\mu,k+1,p}(f)}{(k-\nu_{n-1})!(\nu_{n-1}-\nu_{n-2})!\cdots(\nu_2-\nu_1)!(\nu_1-\mu)!\mu!},$$

$$S_{\mu,k+1,p}(f)=\sum_{i=0}^{\mu-1}\Delta^{\nu_1+1,\nu_2-\nu_1,\dots,\nu_{n-1}-\nu_{n-2},\,k-\nu_{n-1}}_{u_1/2^{p+1},\,u_2,\dots,u_n}f\left(x_1+\frac{iu_1}{2^{p+1}},x_2,\dots,x_n\right).$$

In the case of functions of two variables, identities (4) have the form:

$$\sum_{\nu=0}^{k}(-1)^{\nu}2^{(k-\nu)l}\frac{k!}{(k-\nu)!\nu!}\Delta^{\nu,k-\nu}_{u_1,u_2}f(x_1,x_2)=$$

$$=2^{kl}\Delta^{k}_{-u_1/2^l,u_2}f\left(x_1+\frac{ku_1}{2^l},x_2\right)+A_{k+1,l}(f), \tag{5}$$

where $A_{k+1,0}(f)=0$, and for $l\geq 1$

$$A_{k+1,l}(f)=\sum_{\nu=0}^{k}(-1)^{\nu}\sum_{\mu=0}^{\nu}\sum_{p=0}^{l-1}\frac{2^{(k-\nu)l+p\nu}k!}{(k-\nu)!(\nu-\mu)!\mu!}\sum_{i=0}^{\mu-1}\Delta^{\nu+1,k-\nu}_{u_1/2^{p+1},u_2}f\left(x_1+\frac{iu_1}{2^{p+1}},x_2\right).$$

From (5) one can obtain a number of identities for partial derivatives and directional derivatives.

Corollary. If $f(x,y)$ has continuous derivatives $f_{x^{\nu},y^{k-\nu}}(x_0,y_0)$ ($\nu=0,1,2,\dots,k$) at some point $M_0(x_0,y_0)$, then for this function at the point M_0 there exists the derivative $f^{(k)}_{\lambda}(x_0,y_0)$ of order k in the direction $y=\lambda x$, and

$$f^{(k)}_{\lambda}(x_0,y_0)=(\lambda^2+1)^{-k/2}\sum_{\nu=0}^{k}\binom{k}{\nu}\lambda^{k-\nu}f^{(k)}_{x^{\nu},y^{k-\nu}}(x_0,y_0). \tag{6}$$

From the identities (4) for $l=0$ one can obtain relations analogous to (6) also for the case of functions of any number of variables.

Using the system (5) for $l=0,1,2,\dots,k$ and observing that the determinant of this system

$$\delta = \prod_{\nu=0}^{k} (-1)^{\nu} \frac{k!}{(k-\nu)!\nu!} \begin{vmatrix} 1 & 1 & 1 & \dots 1 & 1 \\ 2^{k} & 2^{k-1} & 2^{k-2} & \dots 2 & 1 \\ 2^{2k} & 2^{2(k-1)} & 2^{2(k-2)} & \dots 2^{2} & 1 \\ . & . & . & . & . & . \\ 2^{k^2} & 2^{k(k-1)} & 2^{k(k-2)} & \dots 2^{k} & 1 \end{vmatrix}$$

is not equal to zero, one can obtain identities expressing each mixed difference $\Delta_{u,v}^{\nu,k-\nu} f(x,y)$ in terms of complete differences of order k, defined (for $n = 2$) by equality (1), and with their help arrive at the validity of the following propositions.

Theorem 2. If at some point $M_0(x_0, y_0)$ the function $f(x,y)$ has continuous mixed derivatives $f_{x^\nu, y^{k-\nu}}(x_0, y_0)$ $(\nu = 0, 1, 2, \dots, k)$, then one can indicate $k+1$ directions $y = \lambda_i x$ $(i = 0, 1, 2, \dots, k)$, and numbers $\mu_i(\lambda_i, k, \nu)$ such that for any ν $(\nu = 1, 2, \dots, k-1)$

$$f^{k}_{x^\nu, y^{k-\nu}}(x_0, y_0) = \sum_{i=0}^{k} \mu_i(\lambda_i, k, \nu) f^{(k)}_{\lambda_i}(x_0, y_0), \tag{7}$$

where $f^{(k)}_{\lambda}(x_0, y_0)$ is the derivative of order k in the direction $y = \lambda_i x$, respectively.

Thus, for example, when $k = 3$,

$$f^{(3)}_{x,y^2}(x_0, y_0) = \frac{\sqrt{2}}{3} f^{(3)}_{\lambda_1}(x_0, y_0) - \frac{\sqrt{2}}{3} f^{(3)}_{\lambda_2}(x_0, y_0) - \frac{1}{3} f^{(3)}_{\lambda_3}(x_0, y_0), \tag{8}$$

where $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 0$.

Theorem 3. If the function $f(x_1, \dots, x_n)$ is continuous in some domain G of the n-dimensional Euclidean space R_n, then for every domain Q such that $\overline{Q} \subset G$, the inequality

$$\omega_{\substack{r_1,\dots,r_n \\ (r_1+r_2+\dots+r_n=k)}} (f; |u_1|, \dots, |u_n|) \leq c_k \omega_k(f; |u_1|, \dots, |u_n|) \tag{9}$$

holds, where

$$\omega_k(f; |u_1|, \dots, |u_n|) = \sup_{M \in \overline{Q}} \sup_{\substack{|t_i| \leq |u_i| \\ (i=1,2,\dots,n)}} \left| \Delta^{k}_{t_1,\dots,t_n} f(x_1, \dots, x_n) \right| \tag{10}$$

is the complete modulus of smoothness of the function $f(x_1, \dots, x_n)$ in G, and

$$\omega_{r_1,\dots,r_n}(f;|u_1|,\dots,|u_n|) = \sup_{M\in\overline{Q}} \sup_{\substack{|t_i|\le|u_i|\\(i=1,2,\dots,n)}} \left|\Delta^{r_1,\dots,r_n}_{t_1,\dots,t_n} f(x_1,\dots,x_n)\right| \tag{11}$$

are its mixed moduli of smoothness, under the assumption that the points $M_u(x_1+ku_1,\dots,x_n+ku_n)\in G$. The constant c_k in the inequality (8) does not depend on the function f, nor on the domains G and Q.

Let us give the values of the constants c_k thus obtained for some k when $n=2$ (see (1)).

If $k=2$, $r_1=1$, $r_2=1$, then $c_2=3/2$*; for $k=3$, $r_1=1$, $r_2=2$, the constant $c_3=5/3$; for the case $k=4$, $r_1=2$, $r_2=2$, the constant $c_4=11/6$; if, however, $k=4$, and $r_1=1$, $r_2=3$, then $c_4=22/7$.

Let us note that, in this way, the identities presented above contain a direct proof of the proposition, stated in 1957 by A. F. Timan, on the validity of inequality (8). For $u_1=u_2=\dots=u_n$, this inequality, as noted in the abstract of Yu. A. Brudnyi (see (5), p. 26), can be obtained by applying the apparatus of the theory of approximation of functions of many variables by so-called quasipolynomials.

From (8) and (4), for $l=0$, it follows that

Theorem 4. *If $f(x_1,\dots,x_n)$ is a continuous function in some domain G, then in every domain Q such that $\overline{Q}\subset G$, the following order relation holds**:*

$$\omega_k\left(f;\ |u_1|,\dots,|u_n|\right) \asymp \max_{\substack{r_1,\dots,r_n\\(r_1+r_2+\dots+r_n=k)}} \left\{\omega_{r_1,\dots,r_n}\left(f;\ |u_1|,\dots,|u_n|\right)\right\}. \tag{12}$$

From the same considerations, based on identity (4), there also follows a theorem analogous to Theorem 4 for the full and mixed moduli of smoothness in the integral metrics L_p, defined in the usual way.

In the periodic case, for $1<p<\infty$, this assertion also follows from a result obtained by the author in (2) (see also (3), p. 21, Theorem 4), with the aid of considerations from the theory of approximation of functions.

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* For $n = 2$ and $k = 2$, inequality (9) with the constant $c_2 = 2$ is given in (4), p. 180.

** $\alpha \asymp \beta$ means that $c_1\beta \leq \alpha \leq c_2\beta$, where c_1 and c_2 are some positive constants.

Note: Figure translations are in progress. See original paper for figures.