# ON SEQUENCES OF LINEAR AGGREGATES OF FUNCTIONS OF MANY COMPLEX VARIABLES

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON SEQUENCES OF LINEAR AGGREGATES OF FUNCTIONS OF MANY COMPLEX VARIABLES

*(Presented by Academician I. M. Vinogradov on 30 VI 1967)*

A. F. Leont' ev posed the following problem. Suppose, for example, that a system of entire functions $\{f_i(z)\}$ is given; using it, form a sequence of linear aggregates

$$P_n(z) = \sum_{i=}^{n} d_i^{(n)} f_i(z) \quad (n = 1, 2, \dots).$$

It is known a priori that this sequence converges uniformly inside a certain domain. It is required to find out what properties the limiting function of the sequence under consideration has. A number of works are devoted to the solution of this problem (see ([1,2]) and others). In the present note we give results of an investigation of the indicated problem as applied to linear aggregates consisting of a system of entire functions of many variables.

Let

$$f(z_1, \dots, z_n) = \sum_{k_1+\dots+k_n=0}^{\infty} a_{k_1\dots k_n} z_1^{k_1} \cdots z_n^{k_n} \tag{1}$$

be an entire function of finite order with respect to the aggregate of variables; let $(\sigma_1, \dots, \sigma_n)$ be the system of conjugate types for the order $(\rho_1, \dots, \rho_n)$ of this function.

We assume that the coefficients of the function (1) satisfy the conditions:

1) $a_{k_1\dots k_n} \neq 0$ $(k_1, \dots, k_n = 0, 1, \dots)$.
2)

$$\lim_{k_1+\dots+k_n\to\infty} \left[ |a_{k_1\dots k_n}| \left(\frac{k_1}{\sigma_1 e \rho_1}\right)^{k_1/\rho_1} \cdots \left(\frac{k_n}{\sigma_n e \rho_n}\right)^{k_n/\rho_n} \right]^{1/(k_1+\dots+k_n)} = b > 0.$$

Let $\{\lambda_j^{(i)}\}$ $(i=1,\dots,n;\ j=1,2,\dots)$ be sequences of complex numbers satisfying the conditions

$$0<|\lambda_1^{(i)}|\le|\lambda_2^{(i)}|\le\cdots,\qquad \overline{\lim_{n\to\infty}}\frac{n}{|\lambda_n^{(i)}|^{\rho_i}}=\tau_i<\infty\quad(i=1,\dots,n).\tag{2}$$

Form the sequence

$$\mathfrak{P}_{m_1,\dots,m_n}(z_1,\dots,z_n)=\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n}d_{i_1\dots i_n}^{(m_1,\dots,m_n)}f(\lambda_{i_1}^{(1)}z_1,\dots,\lambda_{i_n}^{(n)}z_n).\tag{3}$$

To investigate the sequence (3), we apply linear operators which are constructed as follows. Let $\varphi(z_1,\dots,z_n)$ be an entire function of finite order with respect to the aggregate of variables, and let $(\chi_1,\dots,\chi_n)$ be the system of its conjugate types for the order of growth $(\rho_1,\dots,\rho_n)$. Construct the function

$$\psi(z_1,\dots,z_n,t_1,\dots,t_n)=\sum_{k_1,\dots,k_n=0}^{\infty}\frac{A_{k_1\dots k_n}(z_1,\dots,z_n)}{t_1^{k_1+1}\cdots t_n^{k_n+1}},\tag{4}$$

where

$$A_{k_1\dots k_n}(z_1,\dots,z_n)=B_{k_1\dots k_n}(z_1,\dots,z_n)/a_{k_1\dots k_n},$$

$$\varphi(\lambda_1,\dots,\lambda_n)f(\lambda_1z_1,\dots,\lambda_{nz}n)=\sum_{k_1,\dots,k_n=0}^{\infty}B_{k_1\dots k_n}(z_1,\dots,z_n)\lambda_1^{k_1}\cdots\lambda_n^{k_n}.$$

Let us note that for $|z_i|\le R_i$ $(i=1,\dots,n)$ the function (4) is a regular function in the domain

$$D:\quad\{\,|t_i|>\mu_i(R_i)=\left[\varkappa_i/b^{\rho_i}\sigma_i+(R_i/b)^{\rho_i}\right]^{1/\rho_i}\quad(i=1,\dots,n)\,\}.$$

Let $F(z_1,\dots,z_n)$ be a function regular in some complete $n$-circular domain $G$ with center at the origin. Introduce the operator

$$\mathcal{L}[F]\equiv\frac{1}{(2\pi i)^n}\int_\Delta\psi(z_1,\dots,z_n,t_1,\dots,t_n)F(t_1,\dots,t_n)\,dt_1\cdots dt_n,\tag{5}$$

where $\Delta$ is the skeleton of the polycylinder $B\{\,|t_i|\le R_i\ (i=1,2,\dots,n)\,\}\in G$.

We show that if the function $F(z_1,\dots,z_n)$ is regular in the complete $n$-circular domain $G$ with center at the origin, containing inside it the polycylinder

$$\mathcal{E}\{\,|z_i| < R_i, \quad R_i > \mu_i(0) \quad (i = 1, \dots, n)\,\},$$

then the operator (5) is defined and represents a regular function in the complete $n$-circular domain

$$G_1 = \bigcup B\{\,|z_i| < r_i,\ \mu_i(r_i) = R_i\ ((i = 1, \dots, n)\,\}$$

$$[\,R_i > \mu_i(0) \quad (i = 1, \dots, n)\,].$$

Here the union is considered over all those and only those systems $R_i > \mu_i(0)$ $(i = 1, \dots, n)$ for which the polycylinders $\{\,|z_i| < R_i\ (i = 1, \dots, n)\,\}$ belong to the domain $G$. In particular, if $F(z_1, \dots, z_n)$ is an entire function, then $\mathcal{L}[F]$ also represents an entire function.

The function $\varphi(z_1, \dots, z_n)$ will be called the **characteristic function**, and $f(z_1, \dots, z_n)$ the **generating function** of the operator (5).

Applying the operator $\mathcal{L}[F]$ as the apparatus for investigating sequences of the form (3), we obtain the following propositions.

**Theorem 1.** Let the sequence (3) converge uniformly inside the polycylinder $\mathcal{E}\{\,|z_i| < R_i,\ R_i > \mu_i(0)\ (i = 1, \dots, n)\,\}$; then the limiting function

$$\mathfrak{P}(z_1, \dots, z_n) = \lim_{m_1, \dots, m_n \to \infty} \mathfrak{P}_{m_1 \dots m_n}(z_1, \dots, z_n)$$

in the polycylinder $B\{\,|z_i| < r_i,\ \mu_i(r_i) = R_i\ (i = 1, \dots, n)\,\}$ satisfies the equation $\mathcal{L}_1[P] = 0$, where $\mathcal{L}_1[P]$ is an operator of the form (5) with characteristic function

$$\varphi_1(z_1, \dots, z_n) = \prod_{j=1}^{\infty} \left\{ \left[ 1 - \left( \frac{z_1}{\lambda_j^{(1)}} \right)^m \right] \cdots \left[ 1 - \left( \frac{z_n}{\lambda_j^{(n)}} \right)^m \right] \right\};$$

here $m$ is an integer $> \max(\rho_1, \dots, \rho_n)$.

**Theorem 2.** The system of functions $\{f(\lambda_{i_1}^{(1)} z_1, \dots, \lambda_{i_n}^{(n)} z_n)\}$ $(i_1, \dots, i_n = 1, 2, \dots)$ is not complete in any domain of the space $C^n$ containing inside it the polycylinder $\mathcal{E}_{0,\delta}\{\,|z_i| \le \mu_i(0)\ (i = 1, \dots, n)\,\}$. If $\tau_i = 0$ $(i = 1, \dots, n)$, then this system is not complete in any domain of the space $C^n$ containing the origin.

**Theorem 3.** Under the conditions of Theorem 1 there exist limits of the coefficients of the sequence (3), i.e.

$$\lim_{m_1, \dots, m_n \to \infty} d_{i_1 \dots i_n}^{(m_1 \dots m_n)} = d_{i_1 \dots i_n} \quad (i_1, \dots, i_n = 1, 2, \dots).$$

**Theorem 4.** Two sequences of linear aggregates (3) and (6)

$$\mathfrak{P}'_{m_1\dots m_n}(z_1,\dots,z_n)=\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} b_{i_1\dots i_n}^{(m_1\dots m_n)} f(\lambda_{i_1}^{(1)}z_1,\dots,\lambda_{i_n}^{(n)}z_n), \tag{6}$$

uniformly convergent in the polycylinder $\mathcal{E}$, converge to one and the same function if and only if the limiting values of the corresponding coefficients are equal.

From this theorem it follows, in particular, that the limiting values of the coefficients of the sequence (3) uniquely determine the limiting function, i.e., to each limiting function $\mathcal{P}(z_1,\dots,z_n)$ there corresponds a definite series of the form

$$\sum_{i_1,\dots,i_n=1}^{\infty} d_{i_1\dots i_n} f(\lambda_{i_1}^{(1)}z_1,\dots,\lambda_{i_n}^{(n)}z_n). \tag{7}$$

The series (7), as examples show, may diverge, and even everywhere. The following theorem shows how it can be summed.

**Theorem 5.** If the sequence (3) converges uniformly in the polycylinder $\mathcal{E}$, and the limiting function $P(z_1,\dots,z_n)$ is regular in the complete $n$-circular domain $G$ with center at the origin, containing the polycylinder $\mathcal{E}$, then inside the domain

$$G_1=\bigcup B\{|z_i|<r_i,\quad \mu_i(r_i)=R_i\quad (i=1,\dots,n)\},$$
$$\{|z_i|<R_i,\ R_i>\mu_i(0)\quad (i=1,\dots,n)\}\Subset G,$$

uniformly

$$\mathcal{P}(z_1,\dots,z_n)=\lim_{p_1,\dots,p_n\to\infty} Q_{p_1\dots p_n}(z_1,\dots,z_n),$$

where

$$Q_{p_1\dots p_n}(z_1,\dots,z_n)=\sum_{i_1,\dots,i_n=1}^{p_1-1,\dots,p_n-1} d_{i_1\dots i_n}\,\Phi_{(p)}(\lambda_{i_1}^{(1)},\dots,\lambda_{i_n}^{(n)}) f(\lambda_{i_1}^{(1)}z_1,\dots,\lambda_{i_n}^{(n)}z_n),$$

$$\Phi_{(p)}(z_1,\dots,z_n)=\prod_{\substack{i_1=p_1\\ \dots\\ i_n=p_n}}\left\{\prod_{k=1}^{n}\left[1-\left(\frac{z_k}{\lambda_{i_k}^{(k)}}\right)^m\right]\right\}\qquad (p_1,\dots,p_n=1,2,\dots).$$

To clarify the question of where and under what conditions the series (7) converges, put

$$\delta_i = \overline{\lim_{k\to\infty}} \frac{1}{|\lambda_k^{(i)}|^{\rho_i}} \ln\left|\frac{1}{\theta_i'(\lambda_k^{(i)})}\right|, \qquad \theta_i(z_i) = \prod_{\nu=1}^{\infty}\left[1-\left(\frac{z_i}{\lambda_\nu^{(i)}}\right)^m\right],$$

$$\overline{\mu}_i(r) = \left[\frac{r_i+\delta_i^+}{b^{\rho_i}\sigma_i} + \left(\frac{r}{b}\right)^{\rho_i}\right]^{1/\rho_i} \qquad (i=1,\dots,n),$$

where $\delta_i^+ = \delta_i$, if $\delta_i > 0$; $\delta_i^+ = 0$, if $\delta_i \le 0$.

**Theorem 6.** Let $\delta_i < \infty$ $(i=1,\dots,n)$, let the sequence (3) converge uniformly in the polycylinder

$$\mathcal{E}_1\{|z_i| < R_i, \quad R_i > \overline{\mu}_i(0) \quad (i=1,\dots,n)\},$$

and let the limiting function $P(z_1,\dots,z_n)$ be regular in the complete $n$-circular domain $G$, containing the polycylinder $\mathcal{E}_1$. Then the series (7) converges absolutely and uniformly inside the domain

$$G_2 = \bigcup B\{|z_i| < r_i,\ \overline{\mu}_i(r_i) = R_i \quad (i=1,\dots,n)\},$$

$$\{|z_i| < R_i,\ R_i > \overline{\mu}_i(0) \quad (i=1,\dots,n)\} \Subset G.$$

In those cases when $\delta_i > 0$ or, equivalently, $\overline{\mu}_i(0) > \mu_i(0)$, no conclusion can be derived from Theorem 6 if the sequence (3) converges only in the polycylinder $\mathcal{E}$. In the indicated cases one may use the following theorem.

**Theorem 7.** If the sequence (3) converges uniformly in the polycylinder $\mathcal{E}$, and the limiting function is regular in the complete $n$-circular domain $G$ with center at the origin, containing within it the polycylinder $\mathcal{E}_1$, then the series (7) converges absolutely and uniformly inside the domain $G_2$.

Let us note some consequences of the last theorem.

**Corollary 1.** Suppose the sequence (3) converges uniformly in the polycylinder $\mathcal{E}$, and the limiting function $\mathcal{F}(z_1,\dots,z_n)$ is entire. If the sequences $\{\lambda_k^{(i)}\}$ $(i=1,\dots,n)$ satisfy the condition $\delta_i < \infty$ $(i=1,\dots,n)$, then the series (7) converges absolutely and uniformly inside the whole space $C^n$.

**Corollary 2.** Suppose the coefficients of the function (1) satisfy the condition

$$\lim_{k_1+\dots+k_n\to\infty}\left[\left|a_{k_1\dots k_n}\right|\left(\frac{k_1}{\sigma_1 e\rho_1}\right)^{k_1/\rho_1}\cdots\left(\frac{k_n}{\sigma_n e\rho_n}\right)^{k_n/\rho_n}\right]^{1/(k_1+\dots+k_n)} = 1.$$

If the sequence (3) converges uniformly in some neighborhood of the origin, and moreover

$$\lim_{k\to\infty} \frac{k}{|\lambda_k^{(i)}|^{\rho_i}} = 0, \qquad \delta_i = 0 \quad (i = 1, \dots, n),$$

then the series (7) converges absolutely and uniformly to the limiting function $\mathcal{F}(z_1, \dots, z_n)$ inside the largest complete $n$-circular domain with center at the origin in which the function $\mathcal{F}(z_1, \dots, z_n)$ is regular.

In conclusion, let us note that from the stated propositions, as a special case $(f(z_1, \dots, z_n) = e^{z_1+\dots+z_n})$, there follows a number of known results concerning multiple series of Dirichlet polynomials (see (3)).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*