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# NORMAL SEQUENCES AND FINITE AUTOMATA

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **NORMAL SEQUENCES AND FINITE AUTOMATA**

*(Presented by Academician P. S. Novikov on 15 V 1967)*

In a number of works <sup>(1-6)</sup> infinite binary sequences that are “random” in one sense or another are studied. The notion of a random sequence considered in <sup>(1-3)</sup> is Wald’s <sup>(2)</sup> refined notion of a collective, introduced by von Mises <sup>(1)</sup>. In <sup>(5,6)</sup> normal sequences are studied. In the present note the connection between these notions is clarified in terms of finite automata.

**1. Collective.** We shall call a *strategy* a predicate defined on the set of all finite binary sequences (including the empty sequence  $\Lambda$ ). A strategy  $f$  defines the following procedure for selecting from an infinite binary sequence  $x_1x_2x_3 \dots$  a subsequence (possibly finite)  $y_1y_2y_3 \dots$ . If  $z_1z_2z_3 \dots$  is the sequence of values of the strategy  $f$  corresponding to the sequence  $x_1x_2x_3 \dots$ :  $z_1 = f(\Lambda)$ ,  $z_2 = f(x_1)$ ,  $z_3 = f(x_1x_2), \dots$ , and  $i_1 < i_2 < i_3 < \dots$  are the numbers of those places for which  $z_{i_k} = 1$ , then  $y_1 = x_{i_1}$ ,  $y_2 = x_{i_2}$ ,  $y_3 = x_{i_3}, \dots$ . The sequence  $x_1x_2x_3 \dots$  is called a collective with respect to a set of strategies  $F$ , containing the strategy identically equal to one, if for all infinite subsequences selected from  $x_1x_2x_3 \dots$  by strategies of the set  $F$ , there exists one and the same limit of the relative frequency of ones

$$p = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i \quad (0 < p < 1).$$

**2. Normal sequence.** An infinite binary sequence  $x_1x_2x_3 \dots$  is called normal if there exists a number  $p$  ( $0 < p < 1$ ) such that for every positive integer  $n$  and every word  $a_1a_2 \dots a_n$  of length  $n$  in the alphabet  $\{0,1\}$ , the following condition is satisfied: in the sequence of words  $x_1x_2 \dots x_n, x_{n+1}x_{n+2} \dots x_{2n}, x_{2n+1}x_{2n+2} \dots x_{3n}, \dots$  the relative frequency of occurrence of the word  $a_1a_2 \dots a_n$  has a limit equal to  $p^k(1-p)^{n-k}$ , where  $k = \sum_{i=1}^n a_i$  is the number of ones in the word  $a_1a_2 \dots a_n$ .

**3. Normal sequence as a collective.** In <sup>(7)</sup> it is established that a sequence is normal if and only if it is a collective with respect to the set of strategies possessing the following property: the strategy selects or does not select the

next symbol of the sequence depending on what the preceding  $l$  symbols are, where for the given strategy the number  $l$  is fixed.

It is clear that for every strategy  $f$  of this kind there exists a finite automaton  $\langle \{0, 1\}, Q, Q^*, \varphi, q \rangle$  with input alphabet  $\{0, 1\}$ , set of states  $Q$ , subset of selected states  $Q^* \subseteq Q$ , transition function  $\varphi : Q \times \{0, 1\} \rightarrow Q$ , and initial state  $q \in Q$ , which computes the strategy  $f$  in the following way:

$$f(x_1 x_2 \dots x_n) = 1 \iff \varphi(\varphi(\dots \varphi(\varphi(q, x_1), x_2) \dots), x_n) \in Q^*.$$

Hence it follows that a collective with respect to the set of strategies computable by finite automata,

is a normal sequence. The natural question arises whether the converse assertion is true. An affirmative answer to it is given by the following

**Theorem.** *A normal sequence is a collective with respect to the set of strategies computable by finite automata.*

The proof of the theorem uses the ergodic property of a finite automaton to whose input a sequence of independent random variables is fed, and the following “nonincrease of measure” property possessed by the transformation defined by a strategy on words. Let the  $p$ -weight of a word  $a_1 \dots a_n$  be the number  $p^l(1-p)^{n-l}$ , where  $l$  is the number of ones in the word  $a_1 \dots a_n$  and  $0 < p < 1$ , and let the  $p$ -weight of a subset of words of fixed length be the sum of the  $p$ -weights of all words in this subset. Then for any strategy  $f$  and any word  $a_1 \dots a_k$ , the set of words of length  $n \geq k$  from which  $f$  selects words beginning with the word  $a_1 \dots a_k$  has  $p$ -weight not exceeding the  $p$ -weight of the word  $a_1 \dots a_k$ .

In conclusion I express my gratitude to Prof. B. A. Trakhtenbrot, who proposed the problem and made a number of valuable comments.

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*Note: Figure translations are in progress. See original paper for figures.*

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