# ON RANDOM PROCESSES GENERATED BY NUMBER-THEORETIC ENDOMORPHISMS

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## Abstract

## Full Text

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*MATHEMATICS*

M. I. GORDIN

# ON RANDOM PROCESSES GENERATED BY NUMBER-THEORETIC ENDOMORPHISMS

*(Presented by Academician Yu. V. Linnik on 19 II 1968)*

1. A. Rényi ([1]) studied transformations of the interval $(0,1)$ defined by the formula $Tx = \{\varphi(x)\}$, where $\{a\}$ denotes the fractional part of the number $a$, and $\varphi = f^{-1}$, with the function $f$ satisfying one of the conditions A, B and condition C. These conditions are as follows:

A. $f$ is defined and decreases on $[1, +\infty)$, $f(1) = 1$, $f$ is strictly positive, continuous and strictly decreasing on $[1, T)$, and $f = 0$ on $[T, +\infty)$, where $T$ is either a natural number or $+\infty$ (in this case it is meant that $\lim_{t\to\infty} f(t) = 0$). In addition, $|f(t_2) - f(t_1)| \le |t_2 - t_1|$, and there exists $\lambda$, $0 < \lambda < 1$, such that $|f(t_2) - f(t_1)| \le \lambda |t_2 - t_1|$ if $1 + f(2) < t_1 < t_2$.

B. $f$ is defined and increasing on $[0, +\infty)$, $f(0) = 0$, $f$ is continuous and strictly increasing on $[0, T]$, and $f = 1$ on $[T, +\infty)$, where $T$ is either a natural number or $+\infty$ (in this case $\lim_{t\to\infty} f(t) = 1$). In addition, $f(t_2) - f(t_1) \le t_2 - t_1$, if $0 \le t_2 < t_1$.

C.

$$\operatorname*{ess\,sup}_{0<x<1} f'_{E_n}(x) \le C \operatorname*{ess\,inf}_{0<x<1} f'_{E_n}(x).$$

Here $E_n = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, where $\varepsilon_i$ are numbers from the range of values of the function $[\varphi(x)]$, $0 < x < 1$ ($[a]$ is the integer part of $a$), $f_{E_n}(x) = f(\varepsilon_1 + f(\varepsilon_2 + \dots + f(\varepsilon_n + x)\cdots))$, and the constant $C$ depends neither on $E_n$ nor on $n$.

It follows from Rényi' s results that if $f$ satisfies one of the conditions A, B and condition C, then the sequence of functions $a(T^n x)$, $n \ge 0$, $a(x) = [\varphi(x)]$, generates on $(0,1)$ the entire Borel $\sigma$-algebra, and for the transformation T on $(0,1)$ there is an invariant measure $\mathbf{P}$, having density $p(x)$ with respect to Lebesgue measure, with

$$1/C \le p(x) \le C$$

almost everywhere on $(0,1)$.

V. A. Rokhlin ([2]) proved that under these conditions the transformation T of the space $X = (0,1)$ with measure $\mathbf{P}$ is an exact endomorphism, i.e. that

$$\bigcap_{k=0}^{\infty} M_k = N.$$

Here $M_0 = M$ is the $\sigma$-algebra of Lebesgue subsets of $(0,1)$;

$$M_k = \mathrm{T}^{-k}(M)$$

is the $\sigma$-algebra of sets differing only by a set of measure 0 from the complete inverse image of some Lebesgue set with respect to $T^k$; $N$ is the $\sigma$-algebra containing only sets of measure 0 or 1. Hence follows the ergodicity of T, proved in ([1]).

The equality

$$\int_A Vg(x)\,\mathbf{P}(dx) = \int_{T^{-1}(A)} g(x)\,\mathbf{P}(dx), \qquad A \in M,$$

defines a linear operator $V$, acting in each of the spaces $L_p(\mathbf{P})$ $(1 \le p \le \infty)$ with norm 1. The operator $V$, which can be defined for any measure-preserving transformation, has a number of interesting properties. In particular, if T is an exact endomorphism of a space–

space $X$ with measure $\mathbf{P}$, then for $g \in L_p(\mathbf{P})$, $1 \le p < \infty$,

$$V^n g \xrightarrow[L_p(\mathbf{P})]{} \int_X g(x)\mathbf{P}(dx).$$

This relation, in the situation considered by us, can be sharpened.

Let us formulate conditions D and E:

D. All the functions $f'_{E_n}$ have bounded variation and

$$\sum_{E_n} \mathrm{var}(f'_{E_n}) \le K,$$

where $K$ does not depend on $n$.

E.

$$\mathrm{var}(f'_{E_n}) \le K \int_0^1 |f'_{E_n}(x)|\,dx,$$

where $K$ does not depend on $E_n$ or $n$.

Condition E is stronger than condition D, since

$$\sum_{E_n} \int_0^1 |f'_{E_n}(x)|\, dx = 1.$$

In Theorems 1, 2, 3 (see below) it is assumed that $f$ satisfies one of conditions A, B and condition C. By $\alpha(n)$ is denoted a function of the form $Ae^{-\lambda n}$, where the constants $A > 0$, $\lambda > 0$ depend only on $f$.

**Theorem 1.** *Let $g$ be a function of bounded variation. If condition D is satisfied, then*

$$\operatorname*{ess\,sup}_{0<x<1} \left| V^n g(x) - \int_0^1 g(u)\mathbf{P}(du) \right| \le \alpha(n)\,\mathrm{var}(g).$$

Let us now put $X_k(x) = [\varphi(T^k x)]$ and denote by $M_a^b$ the $\sigma$-algebra generated by all the functions $X_k$, $0 \le a \le k \le b \le \infty$.

**Theorem 2.** *Let condition D be satisfied. Then*

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \le \alpha(n)\mathbf{P}(B),$$

if $A \in M_0^k$, $B \in M_{k+n}^\infty$, $n \ge 1$.

**Theorem 3.** *Let condition E be satisfied. Then*

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \le \alpha(n)\mathbf{P}(A)\mathbf{P}(B),$$

if $A \in M_0^k$, $B \in M_{k+n}^\infty$, $n \ge 1$.

**Remark 1.** Condition E is satisfied if, for example, $f''$ exists and for some $p > 1$

$$\sup_l \int_0^1 \left| \frac{f''(x+l)}{f'(x+l)} \right|^p dx < +\infty.$$

The numbers $l$ run through the range of values of the function $[\varphi(x)]$, $x \in (0,1)$.

**Remark 2.** If $\theta > 1$, $\theta$ nonintegral, then $f(x) = x/\theta$ does not satisfy condition B. However, as shown in ([1]) and ([3]), the sequence $X_k$ generates the Borel $\sigma$-algebra, and the transformation $T$ has an invariant measure. For the case $\theta > 2$ one can prove analogues of Theorems 1 and 2 with $\alpha(n) = A(2/\theta)^n$.

**Remark 3.** The results of Theorems 1, 3 for $f(x) = 1/x$ give estimates known in the metric theory of continued fractions ([8]). Under more stringent restrictions on $f$, an estimate of order $Ae^{-\lambda\sqrt{n}}$ was obtained in ([4]).

**2.** Here probabilistic notation will be used. All mathematical expectations are taken with respect to the invariant measure $\mathbf{P}$. Let $g \in L_2(\mathbf{P})$, $\mathbf{E}g = 0$. The sequence $g(T^k x)$, $k \geq 0$, forms a stationary random process. A number of results are known which make it possible to apply limit theorems to similar processes

probability theory, provided only that these processes are asymptotically independent in a certain sense, as, for example, the process $X_k$ in Theorem 2 (for the central limit theorem see, for example, (5), Ch. XVIII). However, one of the conditions for the applicability of such theorems is the relation

$$B_n(g) = \mathbf{E}\left(\sum_{k=0}^{n-1} g(T^k x)\right)^2 \xrightarrow[n\to\infty]{} \infty,$$

whose verification is not always simple.

**Theorem 4.** *If $f$ satisfies the conditions of Theorem 1 and $f'$ is continuous, $g(x) = G([\varphi(x)])$, $\mathbf{E}g = 0$, $\mathbf{E}g^2 < \infty$, then*

$$B_n(g) \xrightarrow[n\to\infty]{} \infty.$$

**Theorem 5.** *If $f$ satisfies the conditions of Theorem 1, $g$ is an unbounded function, $Vg$ has bounded variation, $\mathbf{E}g = 0$, $\mathbf{E}g^2 < \infty$, then*

$$B_n(g) \xrightarrow[n\to\infty]{} \infty.$$

Let $q_n(x)$ be the denominator of the $n$-th convergent of the expansion of the number $x$ into an ordinary continued fraction. With the aid of Theorem 5, for $f(x) = 1/x$,

$$g(x) = \ln\frac{1}{x} - \int_0^1 \ln\frac{1}{x}\, p(x)\, dx, \quad \text{where } p(x) = \frac{1}{(1+x)\ln 2},$$

one can prove that

$$\int_0^1 \left(\ln q_n(x) - \int_0^1 \ln q_n(u)\, p(u)\, du\right)^2 p(x)\, dx \xrightarrow[n\to\infty]{} \infty.$$

Relying on this relation, one can prove the central limit theorem and the law of the iterated logarithm for $\ln q_n$ (see (7)).

**Remark 4.** The result of Theorem 4 for $f(x) = 1/x$ and the above relation for $\ln q_n$ were stated without proof by W. Doeblin (6).

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Leningrad State University
named after A. A. Zhdanov

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*Note: Figure translations are in progress. See original paper for figures.*