APPROXIMATION OF FUNCTIONS OF REAL VARIABLES BY ALGEBRAIC POLYNOMIALS IN A CLOSED DOMAIN

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Abstract

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APPROXIMATION OF FUNCTIONS OF REAL VARIABLES BY ALGEBRAIC POLYNOMIALS IN A CLOSED DOMAIN

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1. As is known ([1,2]), the situation with approximation of functions of one variable on a closed interval is as follows. If a function $f(x) \in C^{k,\alpha}(-1,1)$ (the k-th derivatives satisfy a Lipschitz condition with exponent α on the interval $[-1,1]$), then the order of its approximation by polynomials is estimated by the quantity

$$\frac{1}{n^{k+\alpha}}\left(\sqrt{1-x^2}+\frac{1}{n}\right)^{k+\alpha};$$

roughly speaking, the approximation at the ends of the interval is twice as good as in its middle. Conversely, if the estimate of approximation on the interval is uniform and equal to $n^{-(k+\alpha)}$, then the function belongs to the class $S^{k,\alpha}(-1,1)$, i.e. the derivatives $f^{(i)}(x)$ are continuous for $i < [k/2]$, and for $i \geq [k/2]$ the functions $\psi_i(x) = f^{(i)}(x)(1-x^2)^{i-k/2}$ are continuous, moreover their local modulus of continuity* is estimated by the quantity

$$\omega(\psi_i; x, \delta) \leq M\left(\frac{\delta}{\sqrt{1-x^2}+\sqrt{\delta}}\right)^{\alpha}.$$

Roughly speaking, the structural properties of $f(x)$ at the ends of the interval are twice as bad as in its middle. In both cases the stated conditions are necessary and sufficient.

The question naturally arises to what extent these results generalize to functions of two variables defined in a closed domain D. Here, too, two formulations of the problem are possible.

A. Starting from the uniform smoothness of $f(x)$ in the domain D, obtain estimates of approximation improving toward the boundary.

B. Starting from a uniform estimate of approximation in D, obtain structural properties worsening toward the boundary (necessary and sufficient for approximation).

In the present paper problem B is considered.

2. For an exact formulation of the problem and the results, let us introduce some classes of functions.

Definition 1. We assign a function $f(x)$ to the class $S_0^{k,\alpha}(a,b)$, if for $k \geq i \geq [k/2]$ the functions

$$\psi_i(f;x) = f(x)(x-a)^{i-k/2} \tag{1}$$

are continuous, for odd k, $\psi_i(f;a) = 0$, and their local modulus of continuity admits the estimate

$$\omega(\psi_i;x,\delta) \leq M_i \left(\frac{\delta}{\sqrt{x-a}+\sqrt{\delta}}\right)^{\alpha},$$

where

$$\omega(g;x,\delta) = \sup_{|h|\leq\delta;\; x,x+h\in[a,b]} |g(x+h)-g(x)|.$$

* The precise definition is given below.

In this case, in $S_0^{k,\alpha}(a,b)$ the norm is introduced

$$\|f\|_{S_0^{k,\alpha}(a,b)} = \max\left\{\max_{1\leq i<k} \|\psi_i\|_C;\ \max_{k/2\leq i\leq k} M_i\right\},$$

where M_i is the least constant for which (1) holds, and ψ_i for $i < k/2$ is equal to $f^{(i)}(x)$.

The connection with the previously introduced class (2) $S^{k,\alpha}(-1,1)$ is obvious: $f \in S^{k,\alpha}(-1,1)$ if and only if, for some δ $(0<\delta<1)$, the functions $f(x)$ and $f(-x)$ belong to $S_0^{k,\alpha}(1,\delta)$.

In what follows, x will denote the point $(x_1,\dots,x_m)$ of the m-dimensional space E_m, and $\bar{x}$ its projection onto the plane $x_1=0$: $\bar{x}=(x_2,\dots,x_m)$. The point x will often be replaced by the pair $(x_1,\bar{x})$. The cube Q_a denotes the set $\{\bar{x} : |x_i| \leq a,\ i=2,\dots,m\}$; the parallelepiped $\Pi_{a,b}$ is the set $\{x : 0 \leq x_1 \leq b,\ \bar{x}\in Q_a\}$, and $K_{a,b}$ is the set $\{x : |x_1| \leq b,\ \bar{x} \in Q_a\}$. $P_n(x)$ always denotes a polynomial of degree $\leq n$ in each of the variables $x_1,\dots,x_m$.

Definition 2. We shall assign a function $f(x)$ to the class $\Sigma_0^{k,\alpha}(\Pi_{a,b})$ if $f(x_1, \bar{x}) \in C^{k,\alpha}(Q_a)$ as a function of $\bar{x}$ for fixed x_1 $(0 \leq x_1 \leq b)$, $f(x_1, \bar{x}) \in S_0^{k,\alpha}(0, b)$ as a function of x_1 for fixed $\bar{x}$, and the quantity

$$\|f\|_{\Sigma_0^{k,\alpha}(\Pi_{a,b})} = \max \left\{ \sup_{\bar{x} \in Q_a} \|f(x_1, \bar{x})\|_{S_0^{k,\alpha}(0,b)}; \sup_{x_1 \in [0,b]} \|f(x_1, \bar{x})\|_{C^{k,\alpha}(Q_a)} \right\}$$

is finite.

Roughly speaking, the structural properties of the function $f(x)$ with respect to the variable x_1 deteriorate twice as much as $x_1 \to 0$, in contrast to the properties with respect to the remaining variables.

Definition 3. A domain D with boundary Γ will be assigned to the class $C^{k,\alpha}$ $(k \geq 1)$ if in some neighborhood of each point $z \in \Gamma$ the equation of the boundary Γ in local coordinates (the x_1-axis is directed along the inward normal to the domain D) has the form $x_1 = \varphi_z(\bar{x})$, where $\varphi_z(\bar{x}) \in C^{k,\alpha}(Q_a)$ for sufficiently small a.

The properties of domains of this class needed below are given by the following lemma.

Lemma 1. If $D \in C^{k,\alpha}$, where $k \geq 1$, then there exist such a, b and $c < b$, independent of z, and a domain D_1, lying strictly inside D, such that:

a) $\varphi_z(\bar{x}) \in C^{k,\alpha}(Q_a)$ for all $z \in \Gamma$, and the quantity

$$\|\Gamma\|_{C^{k,\alpha}} = \sup_{z \in \Gamma} \|\varphi_z\|_{C^{k,\alpha}(Q_a)}$$

is finite;

b) the domain $\Pi_{a,b}^z = \{x : 0 \leq x_1 - \varphi_z(\bar{x}) \leq b,\ \bar{x} \in Q_a\}$ lies in D, while $\Pi_{a,c}^z$ lies outside D_1 for all $z \in \Gamma$;

c) the domains $\Pi_{a,b}^z$ $(z \in \Gamma)$ and D_1 form a closed covering of the domain D.

Remark. If $\alpha = 0$, i.e. $\Gamma \in C^k$, then, besides the finiteness of $\|\Gamma\|_{C^k}$, more can be proved: there exists a function $\omega(\varepsilon)$ of modulus-of-continuity type ((1), p. 108) such that

$$\omega(D^\nu \varphi_z; \varepsilon) \leq M\omega(\varepsilon),$$

$$\nu = (\nu_2, \dots, \nu_m), \quad |\nu| = \nu_2 + \dots + \nu_m \leq k, \quad \nu_i \geq 0, \quad D^\nu = \partial^{|\nu|} / \partial x_2^{\nu_2} \cdots \partial x_m^{\nu_m}.$$

Definition 4. A function $f(x)$, defined in a domain $D \in C^{k,\alpha}$, will be assigned to the class $\Sigma^{k,\alpha}(D)$ if $f \in C^{k,\alpha}(D_1)$, $f_z(x) = f(x_1 - \varphi_z(\bar{x}), \bar{x}) \in$

$\in \Sigma_0^{k,\alpha}(\Pi_{a,b})$ for every $z \in \Gamma$ (D_1, a, and b are defined in Lemma 1) and the quantity

$$\|f\|_{\Sigma^{k,\alpha}(D)} = \max\left\{\|f\|_{C^{k,\alpha}(D_1)};\ \sup_{z\in\Gamma}\|f_z\|_{\Sigma_0^{k,\alpha}(\Pi_{a,b})}\right\}$$

is finite.

Roughly speaking, the class $\Sigma^{k,\alpha}(D)$ consists of functions for which, near the boundary, the structural properties in the direction normal to the boundary deteriorate by a factor of two, while the properties on surfaces "parallel" to the boundary Γ do not depend on the distance to it.

3. The main results are contained in Theorems 1 and 2.

Theorem 1. Let $D \in C^{k+p}$, where $p = 2$ for even k and $p = 3$ for odd k, and suppose there exists a sequence of polynomials $P_n(x)$ $(n \geq 1)$ such that

$$\|f - P_n\|_{C(D)} \leq Mn^{-(k+\alpha)}.$$

Then $f \in \Sigma^{k,\alpha}(D)$ and

$$\|f\|_{\Sigma^{k,\alpha}(D)} \leq N_1 M,$$

where N_1 does not depend on f or P_n, but only on the domain D and the numbers k, α.

Theorem 2. Let the domain D have a boundary whose equation has the form $\Phi_l(x) = 0$, where Φ_l is a polynomial of degree $\leq l$, and $\operatorname{grad}\Phi_l(z) \neq 0$ for $z \in \Gamma$.

Then for every function $f(x) \in \Sigma^{k,\alpha}(D)$ one can specify a sequence of polynomials $P_n(x)$ such that

$$\|f - P_n\|_{C(D)} \leq N_2\|f\|_{\Sigma^{k,\alpha}(D)}\, n^{-(k+\alpha)},$$

where N_2 depends only on D, k, l, α, but not on f.

The proof is based on Theorems 3-5, which are also of independent interest.

Theorem 3. In order that the function $f(x)$ belong to $\Sigma_0^{k,\alpha}(\Pi_{a,b})$, it is necessary and sufficient that $F(t_1, \bar{t}) = f(t_1^2, \bar{t})$ belong to $C^{k,\alpha}(K_{a,b})$, and moreover

$$N_3\|f\|_{\Sigma_0^{k,\alpha}(\Pi_{a,b})} \leq \|F\|_{C^{k,\alpha}(K_{a,b})} \leq N_4\|f\|_{\Sigma_0^{k,\alpha}(\Pi_{a,b})},$$

where N_3, N_4 do not depend on f.

Theorem 4 (invariance of $\Sigma^{k,\alpha}(D)$ with respect to smooth transformations). Let $y = \sigma(x)$ be a change of coordinates of class $C^{k,\alpha}(D)$, i.e., one-to-one from D onto $\widetilde{D}$, with the functions $y_i = \sigma_i(x) \in C^{k,\alpha}(D)$, and the functions $x_i = \tau_i(y)$ of the inverse transformation $x = \tau(y)$ belonging to $C^{k,\alpha}(\widetilde{D})$, and

$$\|\sigma_i\|_{C^{k,\alpha}(D)} \leq M, \qquad \|\tau_i\|_{C^{k,\alpha}(\widetilde{D})} \leq M.$$

Then the function $f(x) \in \Sigma^{k,\alpha}(\widetilde{D})$ passes into $\tilde{f}(y) = f(\tau(y))$, belonging to $\Sigma^{k,\alpha}(D)$, and moreover

$$N_5\|f\|_{\Sigma^{k,\alpha}(D)} \leq \|\tilde{f}\|_{\Sigma^{k,\alpha}(\widetilde{D})} \leq N_6\|f\|_{\Sigma^{k,\alpha}(D)},$$

where N_5, N_6 do not depend on f, but only on D, M, k, α.

Theorem 5. If $f \in \Sigma^{k,\alpha}(D)$, $g \in C^{k,\alpha}(D)$, then $fg \in \Sigma^{k,\alpha}(D)$ and

$$\|fg\|_{\Sigma^{k,\alpha}(D)} \leq M\|f\|_{\Sigma^{k,\alpha}(D)},$$

where M depends on g.

In the proof of Theorem 1 the main role is played by

Lemma 2. Let $D \in C^{k+p}$; $P_n(x)$ is a polynomial of degree $\leq n$ in each of the arguments, written in local coordinates (for a point $z \in \Gamma$.

Then there exist such ε, δ that $\Pi^z_{2\delta,2\varepsilon} \subset D$, and for the function $\mathcal{G}_n(x) = P_n(x - \varphi_z(\bar{x}), \bar{x})$, for fixed x_1 $(0 \leq x_1 \leq \delta)$, the estimate

$$\|D^\nu \mathcal{G}_n(\bar{x})\|_{C(Q_\delta)} \leq N_7 n^{|\nu|} \|P_n(x)\|_{C(\Pi^z_{2\delta,2\varepsilon})},$$

is valid, where N_7 does not depend on P_n and z.

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REFERENCES

[1] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
[2] A. L. Fuksman, UMN, 20, no. 4, 187 (1965).

Note: Figure translations are in progress. See original paper for figures.