UNIFORMITIES ON X AND TOPOLOGIES ON $\mathfrak{H}(\mathfrak{H}(X))$

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Abstract

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UNIFORMITIES ON X AND TOPOLOGIES ON $\mathfrak{H}(\mathfrak{H}(X))$

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If u is a uniformity in a topological space X, then in the space $\mathfrak{H}(X)$ of nonempty subsets of X it determines a "Hausdorff uniformity" with a fundamental system of covers of the form

$$\Upsilon = \{\mathfrak{u}\langle A\rangle \mid A \in \mathfrak{H}(X)\},$$

where $\mathfrak{u}$ is an arbitrary uniform cover from u, and

$$\mathfrak{u}\langle A\rangle = \{B \in \mathfrak{H}(A) \mid B \subset \mathrm{St}(A, \mathfrak{u}) \ \&\ A \subset \mathrm{St}(B, \mathfrak{u})\}.$$

Analogously one defines a uniformity in the space $H(X) \subset \mathfrak{H}(X)$ of closed subsets* of X. The latter space is Hausdorff; on the other hand, in the space $\mathfrak{H}(X)$ the closure of a one-element set $\{A\}$, $A \subset X$, consists of all sets $M \in \mathfrak{H}(X)$ having the same closure as A: $[A] = [M]$ (in the sense of X). Obviously, by virtue of the latter condition, the uniformity in $\mathfrak{H}(X)$ is completely determined by the uniformity in $H(X)$.

D. R. Isbell in ([5]) erroneously** asserted that different uniformities in the space X determine, in the indicated way, different topologies in the space of subsets $H(X)$, and that in this sense the space $H(X)$ suffices to distinguish uniformities in X. Contradictory examples were found by A. Ward ([1]), and then by A. A. Ivanov ([2]) and by Isbell himself ([3]). D. H. Smith in ([4]) proved the following weakened version of Isbell' s assertion:

Theorem (D. H. Smith). If the proximities of the uniformities u and v in the space X are distinct, then the topologies they determine in $H(X)$ are also distinct.

It turns out, however, that uniformities in X can nevertheless be distinguished with the aid of the space $\mathfrak{H}(\mathfrak{H}(X))$:

Theorem 1. *If the uniformities u, v of a topological space X are distinct, then the topologies generated by them in the space $\mathfrak{H}(\mathfrak{H}(X))$ (and in $\mathfrak{H}(H(X))$) are also distinct***.*

Taking into account Smith' s theorem cited above, this assertion follows directly from the following result:

Theorem 2. *Let two uniformities u, v, $u \neq v$, be compatible with the topological space X. The proximities of the uniformities u and v, determined by them in the space of closed subsets $H(X)$, are also distinct.*

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* If the space X is metrizable, then $H(X)$ is metrizable by means of the well-known Hausdorff metric.

** Noted by Smith (D. H. Smith). For metric uniformities, Isbell' s assertion is true.

*** Of course, it also follows from this that the topological spaces $H(H(X,u))$ and $H(H(X,v))$ are distinct, but they may already, generally speaking, have noncoinciding sets. It is for this reason that we prefer to formulate the given result for spaces of all subsets.

Proof. Suppose that $\mathfrak{U} \setminus \mathfrak{V} \neq \emptyset$. Choose a cover $\mathcal{U} \in \mathfrak{U}$ for which $\mathcal{U}^{**} \notin \mathfrak{V}$. In the space $H(X)$ consider the sets

$$\mathfrak{A} = \{\{x\} \mid x \in X\}$$

and

$$\mathfrak{B} = \{\{x, y\} \mid x \notin \mathrm{St}(y, \mathcal{U}^*)\}.$$

The cover $\mathcal{U}$ defines in $H(X)$ the cover Υ indicated above; there also exists a cover $\Psi \in \mathfrak{U}$ such that

$$\mathrm{St}(\mathfrak{z}, \Psi) \subset \mathcal{U}_{\{\mathfrak{z}\}} \in \Upsilon,$$

whatever $\mathfrak{z} \in H(X)$.

Now let $\{x, y\} \in \mathrm{St}(\mathfrak{A}, \Psi)$. Hence there exists $\mathfrak{z} \in \mathfrak{A}$ with

$$\{x, y\} \in \mathrm{St}(\mathfrak{z}, \Psi) \subset \mathcal{U}_{\{\mathfrak{z}\}}.$$

Put $\mathfrak{z} = \{z\}$; then

$$\mathcal{U}_{\{\mathfrak{z}\}} = \{C \in H(X) \mid C \subset \mathrm{St}(z, \mathcal{U}) \,\&\, z \in \mathrm{St}(C, \mathcal{U})\} = \{C \in H(X) \mid C \subset \mathrm{St}(z, \mathcal{U})\},$$

since the second factor in the conjunction is, obviously, a consequence of the first. Thus

$$\{x, y\} \subset \mathrm{St}(z, \mathcal{U}).$$

Therefore $x \in \mathrm{St}(y, \mathcal{U}^*)$, and consequently $\{x, y\} \notin \mathfrak{B}$. Thus we have

$$\mathrm{St}(\mathfrak{A}, \Psi) \cap \mathfrak{B} = \emptyset$$

and $\mathfrak{A}\,\delta\,\mathfrak{B}$.

In order to prove $\mathfrak{A}\,\bar{\delta}\,\mathfrak{B}$, assume the contrary. Thus, suppose that there is a cover $\mathfrak{B} \in \mathfrak{V}$ for which

$$\bigcup\{\mathfrak{B}_{\{\mathfrak{z}\}} \mid \mathfrak{z} \in \mathfrak{A}\} \cap \mathfrak{B} = \emptyset.$$

The latter means that, whatever the points x and y in the space X, from the existence of $z(x,y) \in X$ with

$$x, y \in \mathrm{St}(z(x,y), \mathfrak{B})$$

it also follows that

$$x \in \mathrm{St}(y, \mathcal{U}^*).$$

Consider an arbitrary $V \in \mathfrak{B}$ and fix $t \in V$. For any point $y \in V$ we have $y \in \mathrm{St}(t, \mathfrak{B})$ and, of course, $t \in \mathrm{St}(t, \mathfrak{B})$; hence

$$y \in \mathrm{St}(t, \mathcal{U}^*).$$

Thus $V \subset \mathrm{St}(t, \mathcal{U}^*) \in \mathcal{U}^{**}$, and therefore $\mathfrak{B}$ is inscribed in $\mathcal{U}^{**}$, contrary to $\mathcal{U}^{**} \notin \mathfrak{V}$.

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Note: Figure translations are in progress. See original paper for figures.