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Abstract

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MATHEMATICAL PHYSICS

V. P. GACHOK

ON THE DESCRIPTION OF SELF-ADJOINT EXTENSIONS OF FIELD OPERATORS

(Presented by Academician N. N. Bogolyubov on 14 IV 1967)

The question of the self-adjoint properties of field operators has been discussed in a number of works ($^{1-11}$). With the aid of the assumption that the set

$$D_0 = \bigcap_{n \geqslant 1} D(A^n(f_1))$$

is dense in the Hilbert space generated by a quasi-analytic functional, the self-adjointness of the closure of the field operator was proved. At the same time, the simplest models of quantum field theory show that the moments of field operators—the vacuum expectations—have growth stronger than quasi-analytic, and, consequently, field operators admit non-unique self-adjoint extensions. The present work is devoted to the description of these extensions. In our proofs there are many analogies with the classical moment problem ($^{12-16}$) and the theory of difference operators (^{17}).

1. Consider the system

$$\Phi = \sum_{j=0}^{\infty} \oplus \Phi_{4j},$$

where Φ_{4j} is a certain space of basic complex-valued functions of $4j$ variables $x_j = (x_j^0, x_j^1, x_j^2, x_j^3)$. In the system Φ the operations of addition, multiplication, and involution are introduced by analogy with (^{2,7}).

Let $W(f)$ be a functional on Φ, with

$$W(f) = \sum_{n=0}^{\infty} W_n(f_n),$$

where $W_n(f_n) \in \Phi_n'$; Φ_n' is the space conjugate to Φ_n. We shall require of $W(f)$ continuity and $W(f \times f^+) \geqslant 0$. It is assumed that there is no degeneracy.

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Consider in the Hilbert space H_W, generated by the scalar product $\langle f, g\rangle = W(f \times g^{+})$, the operator $A(f_1)$, defined by the equality

$$A(f_1)g = g \times f_1,$$

where

$$f_1 = \{0, f_1(x), 0, 0, ...\}, \qquad f_1(x) = \overline{f_1(x)} \in \Phi_4.$$

The next requirement imposed on $W(f)$ is the condition that

$$D_0 = \bigcap_{n \geqslant 1} D(A^n(f_1))$$

be dense in H_W.

Since D_0 is dense in H_W, there exists an infinite orthonormal sequence of vectors $\{h_n\}_0^\infty \in D_0$ which is generating. Corresponding to this basis, H_W decomposes into the orthogonal sum of subspaces $H_W^{h_n}$,

$$H_W = \sum_{n=0}^{\infty} \oplus H_W^{h_n},$$

where $g \in H_W$, if

$$\sum_{n=0}^{\infty} \|g_n\|^2_{H_W^{h_n}} < \infty, \qquad g_n \in H_W^{h_n}.$$

Since each $H_W^{h_n}$ reduces the operator $E(\lambda; f_1)$—the generalized resolution of the identity of the operator $A(f_1)$—in the case of an infinite interval Parseval' s equality has the form

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f^{(n)}(\lambda)\overline{g^{(n)}(\lambda)}\, d\rho_{h_n}(\lambda; f_1) = (\tilde{f}, \tilde{g})_{\mathcal{L}_2(\infty, d\rho_h(\lambda; f_1))},$$

where

$$\rho_h(\lambda; f_1) = \langle E(\lambda; f_1)h, h\rangle.$$

According to the isometry, between H_n and $\mathcal{L}_2(\infty, d\rho_h(\lambda; f_1))$ the operator $A(f_1)$ corresponds, in each $\widetilde{H}_W^{h_n} = \mathcal{L}_2(\infty, d\rho_{h_n}(\lambda; f_1))$, to the operator $A(f_1)$ of multiplication by λ. If the sequence of vectors $\varphi_0^{(n)} = h_n, \varphi_1^{(n)}, \dots, \varphi_k^{(n)}, \dots$

forms an orthonormal basis in $H_W^{h_n}$, obtained by orthogonalizing the sequence $h_n, A(f_1)h_n, \dots, A^k(f_1)h_n, \dots$, then it corresponds to the orthonormal system of polynomials $P_k^{(n)}(\lambda; f_1)$ in $\widehat{H}_W^{h_n}$, and

$$\varphi_k^{(n)} = P_k^{(n)}(A(f_1))h_n = \int_{-\infty}^{\infty} P_k^{(n)}(\lambda; f_1)\, dE(\lambda; f_1)h_n.$$

Lemma. The operator $A(f_1)$ has equal deficiency indices (m, m), where m is equal to the number of finite constants

$$K_n = \sum_{k=0}^{\infty} |P_k^{(n)}(\lambda; f_1)|^2, \qquad n = 0, 1, \dots .$$

We now proceed to describe all self-adjoint extensions of the operator $A(f_1)$ in H_W in the case where $\rho_h(\lambda; f_1)$ is an orthogonal spectral measure. To this end, let us first consider the operator $A(f_1)$ in $H_W^{h_n}$. Here the Stieltjes transform of the spectral measure $\rho_{h_n}(\lambda; f_1)$ has the form

$$M_{h_n}(z; f_1) = \langle R_{z;f_1} h_n, h_n \rangle = \int_{-\infty}^{\infty} \frac{d\rho_{h_n}(\lambda; f_1)}{\lambda - z}, \tag{1}$$

where $R_{z;f_1}$ is the resolvent of the operator $A(f_1)$.

Denote by $\Gamma_{h_n}(\lambda; f_1)$ the Hamburger-Weyl circle determined by the equality

$$\frac{\overline{M_{h_n}(\lambda; f_1)} - M_{h_n}(\lambda; f_1)}{\overline{\lambda} - \lambda} = \sum_{k=0}^{\infty} \left| Q_k^{(n)}(\lambda; f_1) + M_{h_n}(\lambda; f_1) P_k^{(n)}(\lambda; f_1) \right|^2, \tag{2}$$

where

$$Q_k^{(n)}(\lambda; f_1) = \int_{-\infty}^{\infty} \frac{P_k^{(n)}(t; f_1) - P_k^{(n)}(\lambda; f_1)}{t - \lambda}\, d\rho_{h_n}(t; f_1). \tag{3}$$

When the operator $A(f_1)$ is self-adjoint in H_W^h, the circle $\Gamma_{h_n}(\lambda; f_1)$ degenerates into a point. Taking into account the equality

$$M_h(z; f_1) = \langle R_{z;f_1} h, h \rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\rho_{h_n}(\lambda; f_1)}{\lambda - z}, \tag{4}$$

we arrive at the following conclusion.

Theorem. In the Hilbert space H_W, generated by the functional $W(f)$ in such a way that D_0 is dense in H_W, the operator $A(f_1)$ admits self-adjoint extensions, which are in one-to-one correspondence with the points of the infinite Cartesian product of the Hamburger-Weyl circles $\Gamma_{h_n}(\lambda; f_1)$,

$$\Gamma_h(\lambda; f_1) = \Gamma_{h_0}(\lambda; f_1) \times \Gamma_{h_1}(\lambda; f_1) \times \cdots \times \Gamma_{h_n}(\lambda; f_1) \times \cdots,$$

and this correspondence is realized by means of formulas (1)–(4).

In an analogous way one can also describe extensions with exit from the space H_W. In this case, instead of the circle $\Gamma_{h_n}(\lambda; f_1)$, we obtain the corresponding disk.

II. Let now

$$\Phi_{4j} = \bigoplus \prod_{a=1}^{j} L_0^{(a)}(R^4)$$

be the tensor product of a certain Hilbert space L_0. In this case the action of any operator $A(f_1)$, when $f_1(x)$ ranges over all of L_0, reduces to a countable number of operators $\{A(e_n)\}_1^\infty$, where $\{e_n\}_1^\infty$ is a basis in L_0. Denote $W_n^s(x_1, \dots, x_n) = W_n(x_1, \dots, x_n)$ in the domain of space-like separated vectors. Then, in the Hilbert space generated by the quasi-analytic functional $W^s(f \times g^+)$, the operators $\{A(e_n)\}_1^\infty$ admit a unique extension to a system of commuting self-adjoint operators.

This fact can be used to obtain integral representations that take local commutativity into account. By means of analytic continuation, $W^s(f \times g^+)$ is continued to $W(f \times g^+)$, for which

$$W(f \times g^+) = \sum_{m,n} \int P_{m+n}(\lambda_1, \dots)\, d_{\lambda_1, \lambda_2, \dots} \chi_{m+n}(e_{\gamma_1}, e_{\gamma_2}, \dots; \lambda_1, \lambda_2, \dots),$$

where $\chi_{m+n}(\dots)$ is a positive definite matrix measure defined on the Borel sets of the space R^∞, and

$$P_{m+n}(\lambda_1, \dots) = \sum_{\substack{\alpha_1, \dots, \alpha_j = 1 \\ \beta_1, \dots, \beta_i = 1}}^{\infty} \prod_{j=1}^{m} \prod_{i=1}^{n} \overline{(f^{(j)}, e_j^{(\alpha_j)})}\, (g^{(i)}, e_i^{(\beta_i)})\, \lambda_j^{(\alpha_j)} \lambda_i^{(\beta_i)}.$$

Using the well-known Challen-Williamson representation for the measure $\chi_{m+n}(\dots)$, we find for the weight function $G_{m+n}(-a_{kl})$ (in the notation of ([18]))

$$G_{m+n}(-a_{kl}) = \int P_{m+n}(\lambda_1, \dots)\, d_{\lambda_1, \lambda_2, \dots} Q_{\lambda_1, \dots}(-a_{kl}),$$

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where $Q_{\lambda_1,\ldots}(-a_{kl})$ is a certain positive measure. In the degenerate case the results remain unchanged (see, for example, (^{17}), § 1, item 4, Ch. VIII).

Institute of Theoretical Physics
Academy of Sciences of the Ukrainian SSR

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