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MATHEMATICS

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Abstract

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MATHEMATICS

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ON A CLASS OF OPERATORS WITH SPECTRUM CONCENTRATED AT ZERO

(Presented by Academician I. M. Vinogradov, April 7, 1967)

Let A be a bounded linear operator acting in a separable Hilbert space $\mathfrak{H}$. The operator A is called **completely non-selfadjoint** if there is no subspace invariant with respect to A and A^* in which it induces a selfadjoint operator.

We shall assign a completely non-selfadjoint operator A to the class $\Lambda^{(\exp)}$ if it satisfies the following conditions: 1) A has no spectral points distinct from zero; 2) $\left(A - \frac{1}{\mu}E\right)^{-1}$ is a function of exponential type; 3) A is dissipative, i.e. $A_I = (A - A^*)/2i \geqslant 0$. If, moreover, $\operatorname{sp} A_I < \infty$ ($\operatorname{sp} A_I = \infty$), then we shall assign A to the class $\Lambda^{[\exp]}$ ($\Lambda^{(\exp)}_\infty$).

The class $\Lambda^{[\exp]}$ was studied by M. S. Livshits, M. S. Brodskii, M. G. Krein, I. Ts. Gokhberg, and G. E. Kisilevskii in papers ([1–8]). The present article is devoted to extending some results of M. S. Brodskii and G. E. Kisilevskii to the class $\Lambda^{(\exp)}$.

We note that all operators of the class $\Lambda^{[\exp]}$ are completely continuous, which cannot be said, in general, of operators of the class $\Lambda^{(\exp)}$.

An operator $A \in \Lambda^{(\exp)}$ can be included in the node

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

where $\mathfrak{G}$ is some separable space and K is a bounded linear operator acting from $\mathfrak{G}$ into $\mathfrak{H}$, such that $KK^* = A_I$. The function

$$W(\lambda) = E - 2iK^*(A - \lambda E)^{-1}K$$

is called the **characteristic** function of the node θ. It has the following properties: 1) its values are bounded linear operators acting in $\mathfrak{G}$; 2) $W(1/\mu)$ is an entire function of exponential type; 3) $\lim_{\lambda\to\infty} \|W(\lambda) - E\| = 0$; 4) $W^*(\lambda)W(\lambda) - E \geqslant 0$ ($\operatorname{Im}\lambda > 0$); 5) $W^*(\lambda)W(\lambda) - E = 0$ ($\operatorname{Im}\lambda = 0, \lambda \neq 0$).

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We shall denote the types of growth of the functions $\left(A - \frac{1}{\mu}E\right)^{-1}$ and $W\left(\frac{1}{\mu}\right)$, respectively, by $\sigma[A]$ and $\sigma[W]$. Obviously,

$$\sigma[W] \leqslant \sigma[A]. \tag{1}$$

By $\Omega_{\mathfrak{G}}^{(\exp)}$ we denote the class of all functions possessing properties 1)–5).

Consider the Hilbert space $\widetilde{\mathfrak{L}}_2(0,l)$ $(l < \infty)$ of matrices of the form

$$f(x) = \|f_1(x)\ f_2(x) \ldots\|, \tag{2}$$

where $f_j(x)$ $(j = 1, 2, \ldots)$ are functions measurable on $(0,l)$ satisfying the condition

$$\sum_{j=1}^{\infty} \int_0^l |f_j(x)|^2\, dx < \infty.$$

The scalar product in $\widetilde{\mathfrak{L}}_2(0,l)$ is defi-

we define by the formula

$$(f,g) = \sum_{j=1}^{\infty} \int_0^l f_j(x)\overline{g_j(x)}\, dx \qquad \left(g(x) = \|g_1(x)\ g_2(x) \ldots \| \in \widetilde{\mathcal{L}}_2(0,l)\right).$$

Define in $\mathcal{L}_2(0,l)$ the operator $\tilde{I}_l$, which assigns to the matrix (2) the matrix

$$\tilde{I}_l f(x) = \|I_l f_1(x)\ I_l f_2(x) \ldots \| \qquad \left(I_l f_j(x) = 2i \int_x^l f_j(t)\, dt\right).$$

Theorem 1*. *If the operator T is induced by the operator $\tilde{I}_l$ in one of its invariant subspaces, then $T \in \Lambda^{(\exp)}$ and $\sigma[T] \leq 2l$. Conversely, if $A \in \Lambda^{(\exp)}$ and $\sigma[A] \leq 2l$, then there exists an invariant subspace of the operator $\tilde{I}_l$ in which an operator T, unitarily equivalent to A, is induced.*

Proof. The first assertion of the theorem is trivial. To prove the second, embed A in the simple dissipative node

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

and let $W(\lambda)$ be the characteristic operator-function of this node. For all $\lambda \neq 0$ there exists the operator $W^{-1}(\lambda)$, and the estimate

$$\|W(\lambda)\| \leq e^{2l|\operatorname{Im}\frac{1}{\lambda}|}$$

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holds, from which it follows that $W^{-1}(\lambda)e^{2il/\lambda} \in \Omega_{\mathfrak{H}}^{(\exp)}$. To complete the proof it remains to refer to Theorem 2 of paper ([2]).

Theorem 2. *Every function $W(\lambda) \in \Omega_{\mathfrak{H}}^{(\exp)}$ is characteristic for some node*

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

where $A \in \Lambda^{(\exp)}$ and $\sigma[A] = \sigma[W]$.

It is known that every completely noncontinuous operator has a nontrivial invariant subspace ([10]) (see also ([11]), § 65). It turns out that operators of the class $\Lambda_\infty^{(\exp)}$ also possess this property. Moreover, the following holds.

Theorem 3. *For every operator $A \in \Lambda_\infty^{(\exp)}$ there exists a continuous strictly increasing function $P(x)$ $(0 \le x \le \infty)$ satisfying the following conditions: 1) the values of the function $P(x)$ are orthoprojectors in $\mathfrak{H}$, with $P(0) = 0$, $P(\infty) = E$; 2) all subspaces $P(x)\mathfrak{H}$ are invariant with respect to A;*

$$3) \qquad \operatorname{sp}(P(x)A_I P(x)) = x \qquad (0 \le x \le \infty). \tag{3}$$

Proof. Consider the operator $B = -A^*$. By Theorem 1, the generality of the argument will not be violated if we assume that B is induced by the operator $\widetilde{I}_l$ in some invariant subspace $\mathfrak{H} \subset \widetilde{\mathcal{L}}_2(0,l)$ $(l = \frac{1}{2}\sigma[A])$. In $\widetilde{\mathcal{L}}_2(0,l)$ there exists a subspace $\mathfrak{H}_0$ possessing the following properties: 1) $\mathfrak{H}$ and $\mathfrak{H}_0^\perp = \widetilde{\mathcal{L}}_2(0,l) \ominus \mathfrak{H}_0$ are invariant with respect to $\widetilde{I}_l$; 2) in $\mathfrak{H}_0$ the operator $\widetilde{I}_l$ induces a Volterra operator with a nuclear imaginary component; 3) $\mathfrak{H} \cap \mathfrak{H}_0^\perp \ne \mathfrak{H}$. From Lemma 4 of paper ([4]) it follows that $\mathfrak{H} \cap \mathfrak{H}_0^\perp \ne 0$ and that A induces in its invariant subspace $\mathfrak{H} \ominus (\mathfrak{H} \cap \mathfrak{H}_0^\perp)$ a Volterra operator with nuclear imaginary component. The subsequent arguments differ only insignificantly from those given in Sec. 1 of paper ([3]).

An operator is called **unicellular** if the set of all its invariant subspaces is ordered by inclusion.

Theorem 4. *The class $\Lambda_\infty^{(\exp)}$ contains no unicellular operators. In other words, every unicellular operator of the class $\Lambda^{(\exp)}$ is Volterra and has a nuclear imaginary component.*

The proof follows from Theorem 3 and the criterion for unicellularity of Volterra operators with nuclear imaginary components, established by M. S. Brodskii and G. E. Kisilevskii in the work ([4]).

* For the case $A \in \Lambda^{[\exp]}$ Theorem 1 was proved by G. E. Kisilevskii.

Theorem 5. Let A be a completely non-self-adjoint operator whose spectrum contains no points different from zero. In order that A belong to the class $\Lambda^{(\exp)}$,

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it is necessary and sufficient that there exist a sequence of subspaces invariant with respect to A, $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset ...$, satisfying the following conditions:

1) $\bigcup_{j=1}^{\infty} \mathfrak{H}_j = \mathfrak{H}$;
2) the operators A_j, induced in $\mathfrak{H}_j$, belong to the class $\Lambda^{[\exp]}$;
3) the set of numbers $\sigma[A_j]$ is bounded.

Proof. Necessity follows from Theorem 3. To prove sufficiency, include A and A_j in simple dissipative nodes

$$= \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix} \quad \text{and} \quad \theta_j = \begin{pmatrix} A_j & P_{jK} \\ \mathfrak{H}_j & \mathfrak{G} \end{pmatrix},$$

where P_j is the orthoprojector onto $\mathfrak{H}_j$. Let $W(\lambda)$ and $W_j(\lambda)$ be the characteristic functions of these nodes. Since $\lim_{j\to\infty} W_j(\lambda)f = W(\lambda)f$ $(f \in \mathfrak{G})$ and the set of numbers $\sigma[A_j]$ is bounded, $W(\lambda)$ has finite type of growth. The required result follows from Theorem 2.

The function $P(x)$ $(0 \leq x \leq \infty)$, introduced in Theorem 3, makes it possible to obtain the following generalization of the known theorem on the triangular representation of Volterra operators ([3]) (see also ([11]), §115).

Theorem 6. Every operator $A \in \Lambda^{(\exp)}$ can be represented in the form

$$A = 2i \lim_{x\to\infty} \int_0^x P(t) A_I \, dP(t),$$

where all the integrals exist in the sense of the operator norm, and the passage to the limit is carried out in the sense of strong convergence.

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